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K-Functionals for Besov Spaces

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We characterize the K-functionals for certain pairs of univariate function spaces including (C, W_1^1), $(L_p, B_q^\alpha(L_p))$, $0<q, p\leqslant\infty$ and $(L_p, B_\lambda^\alpha(L_\lambda))$, where $0<p, \alpha<\infty$, and $\lambda:=(\alpha+1/p)^{-1}$. © 1991 Academic Press, Inc.

1. INTRODUCTION

The K-functional was introduced by J. Peetre as a means of generating interpolation spaces. If X_0, X_1 is a pair of quasi-normed spaces which are continuously embedded in a Hausdorff space X, then their K-functional, defined for all $f\in X_0+X_1$, is

$$K(f, t):=K(f, t, X_0, X_1):=\inf_{f=f_0+f_1}(\|f_0\|_{X_0}+t\,\|f_1\|_{X_1}). \tag{1.1}$$

In some cases, the K-functional is defined by using a semi-norm for X_1; we always make clear when this K-functional is intended.

If T is a linear operator which is bounded on X_0 and X_1, then it is easy to see that

$$K(Tf, t, X_0, X_1)\leqslant MK(f, t, X_0, X_1) \tag{1.2}$$

with M depending only on the norms of T on X_0 and X_1. The space $(X, Y)_{\theta,q}$, $0<\theta<1$, $0<q\leqslant\infty$, is the collection of functions $f\in X_0+X_1$ such that

$$|f|_{(X_0, X_1)_{\theta,q}}:=\begin{cases}\left(\int_0^\infty [t^{-\theta}K(f, t)]^q\,\frac{dt}{t}\right)^{1/q}, & 0<q<\infty,\\ \sup_{t\geqslant 0} t^{-\theta}K(f, t), & q=\infty.\end{cases} \tag{1.3}$$

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It follows from (1.2) that $(X_0, X_1)_{\theta,q}$ is an interpolation space for the pair (X_0, X_1); i.e., every linear operator which is bounded on X_0 and X_1 is bounded on $(X_0, X_1)_{\theta,q}$. This method of generating interpolation spaces is called the real method of interpolation.

One of the main problems in interpolation theory is to describe the spaces $(X_0, X_1)_{\theta,q}$ for pairs of classical spaces. While this can sometimes be managed without an explict characterization of the K-functional for the pair, the K-functional provides finer information about interpolation and perhaps more importantly often points to classical quantities which are at the heart of understanding this pair of spaces. For example, the K-functionals for pairs of L_p spaces can be described in terms of rearrangements (see [1]), those for Sobolev spaces in terms of rearrangements of derivatives [7], and so on.

As another example of the characterization of K-functionals which is closely related to the subject of this paper, we consider interpolation for the pair $L_p(I)$, $W_p^r(I)$, where $I = [0, 1]$ and W_p^r is the Sobolev space consisting of all functions $f \in L_p(I)$ which have $(r-1)$ absolutely continuous derivatives and rth derivatives $f^{(r)} \in L_p(I)$. The Sobolev space has the semi-norm $|f|_{W_p^r(I)} := \|f^{(r)}\|_{L_p(I)}$ and norm $\|f\|_{W_p^r(I)} := \|f\|_{L_p(I)} + |f|_{W_p^r(I)}$. In this case, using the semi-norm in the definition of (1.1) we have for $1 \leqslant p \leqslant \infty$, $r = 1, 2, ...$

$$K(f, t^r, L_p, W_p^r) \sim \omega_r(f, t)_p, \tag{1.4}$$

where ω_r is the rth order modulus of smoothness of $f \in L_p$:

$$\omega_r(f, t)_p = \omega_r(f, t, I)_p := \sup_{|h| \leqslant t} \|\Delta_h^r(f, \cdot)\|_{L_p(I_{rh})}. \tag{1.5}$$

Here Δ_h^r is the rth order difference with step h and $I_{rh} = \{x : x, x + rh \in I\}$.

It follows from the characterization (1.4) that

$$(L_p, W_p^r)_{\theta,q} = B_q^{\theta r}(L_p) \tag{1.6}$$

with $B_q^\alpha(L_p)$ the Besov spaces which are defined for $0 < \alpha < r$ and $0 < p, q \leqslant \infty$ as the set of all functions $f \in L_p(I)$ for which

$$|f|_{B_q^\alpha(L_p(I))} := \begin{cases} \left(\int_0^\infty [t^{-\alpha}\omega_r(f, t)_p]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t \geqslant 0} t^{-\alpha}\omega_r(f, t)_p, & q = \infty \end{cases} \tag{1.7}$$

is finite. We define the following "norm" for $B_q^\alpha(L_p(I))$:

$$\|f\|_{B_q^\alpha(L_p(I))} := \|f\|_{L_p(I)} + |f|_{B_q^\alpha(L_p(I))}.$$

Once the K-functional $K(f, t)$ for a pair (X_0, X_1) is known, we can calculate the K-functional for the pair (Y_0, Y_1) for $Y_i := X_{\alpha_i, q_i}$, $i = 0, 1$, from Holmstedt's formula (see [1, p. 307])

$$K(f, t^\lambda; Y_0, Y_1) \sim \left(\int_0^t (s^{-\alpha_0} K(f, s))^{q_0} \frac{ds}{s}\right)^{1/q_0} + t^\lambda \left(\int_t^\infty (s^{-\alpha_1} K(f, s))^{q_1} \frac{ds}{s}\right)^{1/q_1}, \tag{1.8}$$

where $\lambda := \alpha_1 - \alpha_0$.

For example, if $1 \leqslant p \leqslant \infty$, then (1.4), (1.6), and (1.8) give the K-functional for $(L_p, B_q^\alpha(L_p))$ and show that $(L_p, B_q^\alpha(L_p))_{\theta, s} = B_s^{\theta\alpha}(L_p)$ provided $1 \leqslant p \leqslant \infty$. The same characterizations hold for $p < 1$ but must be proved by different techniques (see Section 3) since the Sobolev spaces are not defined for $p < 1$.

Interpolation for the pairs $(L_p, B_q^\alpha(L_\tau))$, where $\tau \neq p$, is more difficult. Little is known about the precise form of the interpolation spaces except for the spacial case $q = (\alpha + 1/p)^{-1}$. We denote the resulting space by $B^\alpha_{p/(p\alpha+1)}$. Then, DeVore and Popov [5] have shown that for $0 < p < \infty$,

$$(L_p, B^\alpha_{p/(p\alpha+1)})_{\theta, p/(p\theta\alpha+1)} = B^{\theta\alpha}_{p/(p\theta\alpha+1)}. \tag{1.9}$$

The same result for Besov spaces defined by Fourier transforms (they correspond to smoothness in H_p) was proved earlier by Peetre [8]. There have been many important applications of (1.9) to various areas of analysis especially nonlinear approximation (see, for example, [5]).

The purpose of the present paper is to shed some light on the nature of the interpolation for Sobolev and Besov spaces by characterizing the K-functional for certain pairs of these spaces. In Section 2, we characterize the K-functional for (C, W_1^1) by using a modified variation of f. This K-functional has important application in approximation by free knot splines. In Section 3, we characterize the K-functional for the pair $(L_p, B_q^\alpha(L_p))$ when $0 < p \leqslant 1$. The characterization is the same as that for $p \geqslant 1$.

Our main results, in Section 5, characterize the K-functional for the pair $(L_p, B^\alpha_{p/(p\alpha+1)})$. For this, we return to the work of Brudnyi [3] and Bergh and Peetre [2] of the 1970s on nonlinear approximation. They characterized the approximation spaces for L_p approximation by splines with free knots as interpolation spaces for the pair $(L_p, V_{\sigma,p})$. Here, $V_{\sigma,p}$, $0 < \sigma < p$, is the collection of functions $f \in L_p$ for which the "variation"

$$|f|_{V_{\sigma,p}} := \sup_{I = \bigcup I_i} \left(\sum_i \omega_r(f, |I_i|, I_i)_p^\sigma\right)^{1/\sigma} \tag{1.10}$$

is finite. Here $r-1$ is the greatest integer in $1/\sigma - 1/p$ and the sup is taken over all partitions $I = \bigcup I_i$.

The results of Brudnyi and Bergh and Peetre were in some sense supplanted by the work of Petrushev [9] and DeVore and Popov [5], who gave similar characterizations for the approximation spaces in terms of the more familiar Besov spaces. However, as we shall see in the present paper, the $V_{\sigma,p}$ spaces and the concept of σ variation are useful for characterizing K-functionals. For example, in Section 4 we characterize $K(f, t, L_p, V_{\sigma,p})$, $0 < p \leqslant \infty$, in terms of local variation and this in turn gives a characterization of the K-functional $K(f, t, L_p, B^\alpha)$. We should mention that, when $p = \infty$, the K-functional for $(C, V_{\sigma,\infty})$ was already computed by Bergh and Peetre [2].

2. The K-Functional for the Pair (C, W_1^1).

Let $f \in C(I)$. For $t > 0$, we denote by π_t partitions of I with $n \leqslant [1/t] + 1$ pieces, that is, $I = \bigcup_{i=1}^n I_i$, where I_i are disjoint subintervals. We define

$$\Omega(f, t) = \sup_{\pi_t} t\left(\sum_{i=1}^{n} \omega(f, |I_i|, I_i)\right),$$

where the sup is taken over all partitions π_t. Here $\omega(f, \cdot, I)$ denotes the modulus of continuity of f on the interval I. Hence Ω is a measure of the variation of f.

THEOREM 2.1. *Let $f \in C(I)$. Then, for $t > 0$, we have*

$$\Omega(f, t) \sim K(f, t, C, W_1^1) := \inf_{g \in W_1^1} \|f - g\|_\infty + t\,\|g'\|_1 \tag{2.1}$$

with absolute constants of equivalency.

Proof. From the definition of $\Omega(f, t)$, it is easy to see that $\Omega(f, t) \leqslant 4\,\|f\|_\infty$ because $nt \leqslant 2$. Since Ω is subadditive (in f), for any $g \in W_1^1$, we have

$$\begin{aligned}\Omega(f, t) &\leqslant \Omega(f - g, t) + \Omega(g, t)\\ &\leqslant 4\,\|f - g\|_\infty + \sup_{\pi_t} t\left(\sum_{i=1}^{n} \int_{I_i} |g'|\right)\\ &= 4\,\|f - g\|_\infty + t\int_I |g'|.\end{aligned}$$

Taking the inf over all $g \in W_1^1$ on the right side of the above inequality, we obtain

$$\Omega(f, t) \leqslant 4K(f, t, C, W_1^1).$$

To reverse this inequality, we fix $t > 0$ and find a balanced partition π_t: $I = \bigcup_{i=1}^n I_i$, $n := [1/t] + 1$, such that

$$\omega(f, |I_i|, I_i) = \omega(f, |I_j|, I_j), \qquad i, j = 1, 2, ..., n. \tag{2.2}$$

To show that such a partition exists, we proceed by induction. We can assume that f is not a constant. There is a balanced partition for $n = 1$. Now suppose that for each $0 < y < 1$ we have a balanced partition of $I_y := [0, y]$ with $n-1$ pieces and let $b_{n-1}(y)$ be the common value in (2.2) for this partition. Then $b_n(y)$ is continuous in y and $b_{n-1}(0) = 0$ and $b_{n-1}(1) > 0$. Therefore, we can choose y such that $b_{n-1}(y) = \omega(f, 1-y, [y, 1])$. If $0 = x_0 < x_1 < \cdots < x_{n-2} < x_{n-1} = y$ is the balanced partition of I_y, then $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n := 1$ provides a balanced partition of $I = [0, 1]$ with n pieces.

Now let g be the continuous piecewise linear function which interpolates f at its breakpoints x_j, $j = 0, 1, ..., n$. If x is any point in $I_j = [x_{j-1}, x_j]$, $j = 1, 2, ..., n$, then $|f(x) - f(x_{j-1})| \leqslant \omega(f, |I_j|, I_j)$ for $x \in I_j$. Hence,

$$\begin{aligned} |f(x) - g(x)| &\leqslant |f(x) - f(x_{j-1})| + \left| \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} \right| |x - x_{j-1}| \\ &\leqslant 2\omega(f, |I_j|, I_j), \qquad x \in I_j,\ j = 1, ..., n. \end{aligned}$$

The function $g \in W_1^1$ and since $f \in C$ and the partition π_t is balanced, we have

$$\begin{aligned} \|f - g\|_\infty (I) &\leqslant \sup_i \|f - g\|_\infty (I_i) \leqslant 2 \sup_i \omega(f, |I_i|, I_i) \\ &\leqslant 2n^{-1} \sum_{i=1}^n \omega(f, |I_i|, I_i) \leqslant 2\Omega(f, t). \end{aligned}$$

Moreover, we have

$$\int_I |g'| = \sum_{i=1}^n \int_{I_i} |g'| = \sum_{i=1}^n |f(x_i) - f(x_{i+1})| \leqslant \sum_{i=1}^n \omega(f, |I_i|, I_i).$$

Hence, we obtain

$$\|f - g\|_\infty (I) + t \int_I |g'| \leqslant 3\Omega(f, t),$$

which gives

$$K(f, t, C, W_1^1) \leqslant 3\Omega(f, t). \quad \blacksquare$$

The same proof also show that

$$K(f, t, C, BV \cap C) \sim \Omega(f, t), \qquad t \geqslant 0, \tag{2.3}$$

where this K-functional is defined using the semi-norm $\mathrm{Var}(f)$ for the space $BV \cap C$. It is well known (see [10, p. 220]) that the error $\sigma_n(f)_\infty$ for approximation in C by piecewise constants with n pieces is related to $K(f, 1/n, C, BV \cap C)$ by direct and inverse inequalities. From these, we obtain

$$\sigma_n(f) = O(n^{-\alpha}) \leftrightarrow \Omega(f, 1/n) = O(n^{-\alpha}), \qquad 0 < \alpha \leqslant 1, n = 1, 2, ... \tag{2.4}$$

We remark that similar statements can be made which characterize the approximation spaces $A_q^\alpha(C)$ (see Section 5).

3. The K-Functional for $(L_p, B_q^\alpha(L_p))$

In this section, we prove the following theorem.

Theorem 3.1. *Let $0 < p, q \leqslant \infty$ and $0 < \alpha < \min(r - 1 + 1/p, r)$. Then, for $f \in L_p(I)$ and $0 < t \leqslant 1$, we have*

$$K(f, t^\alpha, L_p, B_q^\alpha(L_p)) \sim t^\alpha \left(\int_t^\infty [s^{-\alpha} \omega_r(f, s)_p]^q \frac{ds}{s} \right)^{1/q} \tag{3.1}$$

with constants of equivalency depending only on α, p, q.

In the case $1 \leqslant p \leqslant \infty$, this follows from (1.4), (1.6), and Homstedt's formula (1.8). We prove this theorem for the case $0 < p \leqslant 1$ by using some results from DeVore and Popov [6].

Let T_n be the dyadic knot sequence:

$$T_n := \{t_j : 1 \leqslant j < 2^n\}, \qquad t_j := t_j^n := j/2^n, j \in Z.$$

We let $\Pi_n := \Pi_{n,r}$ denote the set of all piecewise polynomials of order r with knots in T_n and let $\mathscr{S}_r(T_n)$ be the space of those functions $S \in \Pi_n$ which are in $C^{r-2}[0, 1]$. If $N(x) := N(x; 0, 1, ..., r)$ is the B-spline of order r whose knots are 0, 1, ..., r then each $S \in \mathscr{S}_r(T_n)$ has the representation

$$S = \sum_j \alpha_j(S) N_{j,n},$$

where $N_{j,n}(x) := N(2^n(x - t_j))$. The coefficient functionals α_j can be extended to all of L_1 (we continue to denote this extension by α_j.) Then, for any $f \in L_1$, we have the well-known quasi-interpolant operators Q_n:

$$Q_n(f) := \sum_j \alpha_j(f)\, N_{j,n}.$$

The operator Q_n is a projector from L_1 onto $\mathcal{S}_r(T_n)$. In particular $Q_n(S)$ is defined for all $S \in \Pi_n$.

Now let $f \in L_p(I)$, $0 < p \leqslant \infty$. We use the quasi-interpolant operators Q_n to generate smooth dyadic splines to approximate f and then to obtain the upper estimates for $K(f, t^\alpha, L_p, B_q^\alpha(L_p))$. We first define a piecewise polynomial $S_n(f) \in \Pi_n$ as

$$S_n(f) := P_{I_j}(x), \qquad x \in I_j := [t_{j-1}, t_j),\ j = 1, ..., 2^n,$$

where P_{I_j} is the best L_p approximation to f from polynomials of degree $< r$ on $[t_{j-r}, t_{j+r}]$. Then we define

$$\bar{Q}_n(f) := Q_n(S_n(f)), \qquad n = 0, 1, ...$$

We denote by

$$s_n(f)_p := \inf_{S \in \mathcal{S}_r(T_n)} \|f - S\|_p,$$

the error of approximation by dyadic splines. Then, in [6], DeVore and Popov have proved the following results.

THEOREM A. *For $f \in L_p(I)$, $0 < p \leqslant \infty$, we have*

$$\|f - \bar{Q}_n(f)\|_p \leqslant C\omega_r(f, 2^{-n})_p, \tag{3.2}$$

where C is independent of f and n.

THEOREM B. *Let $\alpha > 0$ and $0 < p, q \leqslant \infty$. If $\alpha < \min(r - 1 + 1/p, r)$, then for $f \in B_q^\alpha(L_p)$ we have*

$$|f|_{B_q^\alpha(L_p)} \leqslant C\left(\sum_{k=0}^{\infty} [2^{k\alpha} s_k(f)_p]^q\right)^{1/q} \tag{3.3}$$

Proof of Theorem 3.1. We fix $0 < t \leqslant 1$. First we prove that the right side $I(f)$ of (3.1) does not exceed a multiple of the left side $K(f)$. We have

$$I(f) \leqslant t^\alpha \left(\int_t^\infty (s^{-\alpha} \|f\|_p)^q \frac{ds}{s}\right)^{1/q}$$
$$\leqslant \|f\|_p\, t^\alpha \left(\int_t^\infty s^{-\alpha q - 1}\, ds\right)^{1/q} \leqslant C \|f\|_p.$$

Moreover, if $g \in B_q^\alpha(L_p)$, we have

$$I(f) \leqslant C(I(f-g)+I(g)) \leqslant C(\|f-g\|_p + t^\alpha \, |g|_{B_q^\alpha(L_p)}).$$

Here and later we use the fact that $\|\cdot\|_p$ is a quasinorm ($\|f+g\|_p \leqslant C\,\|f\|_p + \|g\|_p$). Taking an inf over all $g \in B_q^\alpha(L_p)$ on the right-hand side of the above inequality, we obtain

$$I(f) \leqslant CK(f).$$

Now we prove the reverse inequality. Since $\omega_r(f, t)_p$ is monotone, we have

$$I(f) \geqslant \omega_r(f, t)_p \, t^\alpha \left(\int_t^\infty s^{-\alpha q - 1} \, ds \right)^{1/q} \geqslant C\omega_r(f, t)_p. \tag{3.4}$$

We let n be the integer such that $2^{-n-1} \leqslant t < 2^{-n}$. For $g = \bar{Q}_n(f)$, we have from Theorem A and (3.4) that

$$\|f-g\|_p \leqslant C\omega_r(f, t)_p \leqslant CI(f). \tag{3.5}$$

On the other hand, by Theorem B and A, we have

$$\begin{aligned} |g|_{B_q^\alpha(L_p)} &\leqslant C\left(\sum_{k=0}^{\infty} [2^{k\alpha} s_k(g)_p]^q \right)^{1/q} = C\left(\sum_{k=0}^{n} [2^{k\alpha} s_k(g)_p]^q \right)^{1/q} \\ &\leqslant C\left(\sum_{k=0}^{n} [2^{k\alpha} s_k(f)_p]^q \right)^{1/q} \leqslant C\left(\sum_{k=0}^{n} [2^{k\alpha} \omega_r(f, 2^{-k})_p]^q \right)^{1/q} \\ &\leqslant C\left(\int_t^\infty [s^{-\alpha} \omega_r(f, s)_p]^q \frac{ds}{s} \right)^{1/q} = Ct^{-\alpha} I(f). \end{aligned}$$

Here, the equality holds because $g \in \mathscr{S}_r(T_n)$ and therefore $s_k(g) = 0$, $k \geqslant n$. Also, the second inequality uses that $s_k(g) \leqslant C(\|f-g\|_p + s_k(f)_p) \leqslant C(s_n(f)_p + s_k(f)_p)$. Now, from the above inequality and (3.5), we obtain

$$K(f, t^\alpha, L_p, B_q^\alpha(L_p)) \leqslant \|f-g\|_p + t^\alpha \, |g|_{B_q^\alpha(L_p)} \leqslant CI(f). \quad \blacksquare$$

4. THE K-FUNCTIONAL FOR $(L_p, V_{\sigma,p})$

We characterize the K-functional for the pair of spaces $(L_p, V_{\sigma,p})$ and then apply this to calculate K functionals for Besov spaces. We first introduce a new kind of modulus of smoothness for $f \in L_p$. Let $0 < \sigma < p$, $\beta := 1/\sigma - 1/p$, and $r := [\beta] + 1$. We define

$$\Omega(f, t)_{\sigma, p} := \sup_{0 < h \leqslant t} \sup_{\pi_h} h^{\beta} \left(\sum_{i=1}^{n} \omega_r(f, |I_i|, I_i)_p^{\sigma} \right)^{1/\sigma}, \tag{4.1}$$

where the second sup is taken over all partitions $\pi_h: I = \bigcup_{i=1}^{n} I_i$ with $n \leqslant [1/h] + 1$.

THEOREM 4.1. *Let $0 < \sigma < p \leqslant \infty$ and $\beta := 1/\sigma - 1/p$. Then for $f \in L_p(I)$ and $t > 0$ we have*

$$K(f, t^{\beta}, L_p, V_{\sigma, p}) \sim \Omega(f, t)_{\sigma, p}.$$

Proof. For $f \in L_p(I)$, by using Hölder's inequality, we have

$$\begin{aligned} \Omega(f, t)_{\sigma, p} &\leqslant \sup_{0 < h \leqslant t} \sup_{\pi_h} h^{\beta} \left(\sum_{i=1}^{n} \|f\|_p^{\sigma} (I_i) \right)^{1/\sigma} \\ &\leqslant C \sup_{0 < h \leqslant t} \sup_{\pi_h} h^{\beta} \left(\sum_{i=1}^{n} \|f\|_p^{p} (I_i) \right)^{1/p} n^{1/\sigma - 1/p} \\ &\leqslant C \|f\|_p (I). \end{aligned} \tag{4.2}$$

Hence, for any $g \in V_{\sigma, p}$, we have

$$\begin{aligned} \Omega(f, t)_{\sigma, p} &\leqslant C(\Omega(f - g, t)_{\sigma, p} + \Omega(g, t)_{\sigma, p}) \\ &\leqslant C(\|f - g\|_p + t^{\beta} |g|_{V_{\sigma, p}}). \end{aligned}$$

We now take an inf over all $g \in V_{\sigma, p}$ on the right-hand side of the last inequality and we obtain

$$\Omega(f, t)_{\sigma, p} \leqslant CK(f, t^{\beta}, L_p, V_{\sigma, p}). \tag{4.3}$$

To prove a converse of this inequality, for $t > 0$ we let $n := [1/t] + 1$. As in the proof of Theorem 2.1, we can find a balanced partition π_t such that

$$\omega_r(f, |I_i|, I_i)_p = \omega_r(f, |I_j|, I_j)_p, \qquad i, j = 1, ..., n.$$

We define

$$g(x) := P_{I_i}(x), \qquad \text{for} \quad x \in I_i,$$

where P_{I_i} are best L_p approximations to f on I_i from polynomials of degree $< r$. Whitney's theorem (see, e.g., [10, p. 195]) gives that $\|f - P_{I_i}\|_p \leqslant C\omega_r(f, |I_i|, I_i)_p$. Since the partition π_t is balanced, we have

$$\|f-g\|_p=\left(\sum_{i=1}^{n}\|f-P_{I_i}\|_p^p(I_i)\right)^{1/p}\leqslant C\left(\sum_{i=1}^{n}\omega_r(f,|I_i|,I_i)_p^p\right)^{1/p}$$
$$=Cn^{1/p}\omega_r(f,|I_i|,I_i)_p=Cn^{1/p-1/\sigma}\left(\sum_{i=1}^{n}\omega_r(f,|I_i|,I_i)_p^\sigma\right)^{1/\sigma}$$
$$\leqslant C\Omega(f,t)_{\sigma,p}. \tag{4.4}$$

Now the function g is a piecewise polynomial of degree $<r$ with n pieces. Hence, for any partition π of I, $I=\bigcup_i I_i'$, we shall have $\omega_r(g,|I_i'|,I_i')_p=0$ if the interval I_i' contains no breakpoints of g. This means that the number of these intervals I_i' which make $\omega_r(g,|I_i'|,I_i')_p\neq 0$ is $\leqslant n$. Hence, in the definition of $|g|_{V_{\sigma,p}}$, we can restrict ourselves to partitions with at most n intervals, i.e., partitions in π_t. Therefore, we have

$$|g|_{V_{\sigma,p}}=\sup_{\pi_t}\left(\sum_{i=1}^{n}\omega_r(g,|I_i'|,I_i')_p^\sigma\right)^{1/\sigma}$$

Now, by (4.2) and (4.4), we obtain

$$|g|_{V_{\sigma,p}}\leqslant C\left\{\sup_{\pi_t}\left(\sum_{i=1}^{n}\omega_r(f,|I_i|,I_i)_p^\sigma\right)^{1/\sigma}\right.$$
$$\left.+\sup_{\pi_t}\left(\sum_{i=1}^{n}\omega_r(f-g,|I_i'|,I_i')_p^\sigma\right)^{1/\sigma}\right\}$$
$$\leqslant C\{t^{-\beta}\Omega(f,t)_{\sigma,p}+t^{-\beta}\Omega(f-g,t)_{\sigma,p}\}$$
$$\leqslant C\{t^{-\beta}\Omega(f,t)_{\sigma,p}+t^{-\beta}\|f-g\|_p\}\leqslant Ct^{-\beta}\Omega(f,t)_{\sigma,p}. \tag{4.5}$$

Then, from (4.4) and (4.5), we obtain

$$K(f,t^\beta,L_p,V_{\sigma,p})\leqslant\|f-g\|_p+t^\beta|g|_{V_{\sigma,p}}\leqslant C\Omega(f,t)_{\sigma,p}. \quad \blacksquare \tag{4.6}$$

5. K-FUNCTIONALS FOR $(L_p, B^\alpha_{p/(p\alpha+1)})$

To characterize the K-functional for these pairs, we use various results which characterize the approximation spaces for free knot spline approximation in terms of interpolation spaces. Let Σ_n denote the class of all piecewise polynomials of degree $<r$ with at most n pieces. For $f\in L_p(I)$, we denote by $\sigma_n(f)_p$ the error of L_p approximation of f by the elements of

Σ_n. Let $\alpha>0$ and $0<q\leqslant\infty$. The approximation space $A_q^\alpha(L_p)$ consists of all $f\in L_p(I)$ such that

$$|f|_{A_q^\alpha(L_p)}:=\begin{cases}\left(\sum_{n=1}^{\infty}[n^\alpha\sigma_n(f)_p]^q\frac{1}{n}\right)^{1/q}, & 0<q<\infty\\ \sup_{n\geqslant 1} n^\alpha\sigma_n(f)_p, & q=\infty\end{cases}$$

is finite. Brudnyi [3] has stated (without proof) that for $0<\sigma<p\leqslant\infty$, $0<q\leqslant\infty$

$$A_q^\alpha(L_p)=(L_p, V_{\sigma,p})_{\alpha/\beta,q} \tag{5.1}$$

provided $\alpha<\beta:=1/\sigma-1/p$ and $r>\beta$. For completeness, we now indicate how to prove (5.1).

According to general results on approximation spaces (see, for example, [5]), it is sufficient to prove the following Jackson and Bernstein inequalities for the pair $(L_p, V_{\sigma,p})$:

$$\text{(i)}\quad \sigma_n(f)_p\leqslant Cn^{-\beta}|f|_{V_{\sigma,p}},\qquad f\in V_{\sigma,p},$$

$$\text{(ii)}\quad |S|_{V_{\sigma,p}}\leqslant Cn^\beta\|S\|_p,\qquad S\in\Sigma_n.$$

Now, (i) follows from the proof of Theorem 4.1. Indeed, in that theorem, we have obtained a free knot spline $g\in\Sigma_n$ which satisfies (4.4):

$$\|f-g\|_p\leqslant C\Omega(f,t)_{\sigma,p},\qquad n=[1/t]+1.$$

Since by the definition of Ω, we have $\Omega(f,t)_{\sigma,p}\leqslant n^{-\beta}|f|_{V_{\sigma,p}}$, (i) follows. Regarding (ii), an argument similar to the derivation of (4.5) gives

$$|S|_{V_{\sigma,p}}=\sup_{\pi_t}\left(\sum_{i=1}^{n}\omega_r(S,|I_i|,I_i)_p^\sigma\right)^{1/\sigma}\leqslant C\sup_{\pi_t} n^\beta\left(\sum_{i=1}^{n}\omega_r(S,|I_i|,I_i)_p^p\right)^{1/p}$$

$$\leqslant Cn^\beta\sup_{\pi_t}\left(\sum_{i=1}^{n}\|S\|_p^p(I_i)\right)^{1/p}\leqslant Cn^\beta\|S\|_p,$$

which is (ii).

Recently, Petrushev [9] has shown that these approximation spaces can also be characterized as interpolation spaces for Besov spaces. Namely, he shows that for $0<p<\infty$, $0<q\leqslant\infty$, and $0<\alpha<\beta$,

$$A_q^\alpha(L_p)=(L_p, B^\beta_{p/(p\beta+1)})_{\alpha/\beta,q} \tag{5.2}$$

holds. Hence, from (5.1) and (5.2), we have

$$(L_p, V_{\sigma,p})_{\alpha/\beta,q}=(L_p, B^\beta_{p/(p\beta+1)})_{\alpha/\beta,q}. \tag{5.3}$$

Moreover, DeVore and Popov [5] have shown that if $0<p<\infty$, $0<\alpha<\beta$, then

$$(L_p, B^{\beta}_{p/(p\beta+1)})_{\alpha/\beta,\lambda}=B^{\alpha}_{p/(p\alpha+1)}, \qquad \text{if} \quad \lambda:=(\alpha+1/p)^{-1}.$$

Therefore, by (5.3), we know that Besov spaces $B^{\alpha}_{p/(p\alpha+1)}$ are the interpolation spaces with respect to the pair of spaces $(L_p, V_{\sigma,p})$,

$$B^{\alpha}_{p/(p\alpha+1)}=(L_p, V_{\sigma,p})_{\alpha/\beta,\lambda},$$

where $0<\alpha<\beta$ and $\lambda:=(\alpha+1/p)^{-1}$. Thus, using Holmstedt's formula (1.8) and Theorem 4.1, we obtain the following result for $K(f, t, L_p, B^{\alpha}_{p/(p\alpha+1)})$.

THEOREM 5.1. *Let $0<p<\infty$, and $\alpha>0$ satisfying $1/\sigma-1/p>\alpha$, then, for $f\in L_p(I)$ and $t>0$, we have*

$$K(f, t^{\alpha}, L_p, B^{\alpha}_{p/(p\alpha+1)})\sim t^{\alpha}\left(\int_t^{\infty}[s^{-\alpha}\Omega(f,s)_{\sigma,p}]^{\lambda}\frac{ds}{s}\right)^{1/\lambda}, \qquad (5.4)$$

where $\lambda:=(\alpha+1/p)^{-1}$.

Also, from Theorem 4.1 and (5.1), we can obtain a characterization of approximation spaces $A^{\alpha}_q(L_p)$ for free knot approximation by splines of order $r>\alpha$:

THEOREM 5.2. *Let $0<p<\infty$, $0<q\leqslant\infty$, and $\alpha>0$. If $\sigma>0$ satisfies $1/\sigma-1/p>\alpha$, then we have*

$$A^{\alpha}_q(L_p)=\left\{f: f\in L_p(I) \text{ such that} \int_0^{\infty}[s^{-\alpha}\Omega(f,s)_{\sigma,p}]^q\frac{ds}{s}<\infty\right\}. \qquad (5.5)$$

COROLLARY 5.3. *Let $0<p<\infty$, and $\alpha>0$. If $\sigma>0$ satisfies $1/\sigma-1/p>\alpha$, then $\sigma_n(f)_p=\mathcal{O}(n^{-\alpha})$ if and only if $\Omega(f,t)_{\sigma,p}=\mathcal{O}(t^{\alpha})$.*

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