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L_p APPROXIMATION BY RECIPROCALS OF TRIGONOMETRIC AND ALGEBRAIC POLYNOMIALS

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ABSTRACT. We give an estimate for the error of L_p approximation by reciprocals of polynomials. These estimates are the analogues of the Jackson and Ditzian - Totik estimates for polynomial approximation.

1. **Introduction.** We are interested in estimating the error of approximation in the L_p norm by reciprocals of polynomials. This is a special case of rational approximation which occurs for example in the study of Padé approximation as the first column in the Padé table. Recently, Leviatan, Levin, and Saff [2] have estimated the error in L_p approximation of $f \in L_{p+1}$ in terms of the modulus of f. Namely they show that the error of such approximation does not exceed $C\,\omega^{\varphi}(f, n^{-1})_{p+1}$ (see §2 for the definition of ω^{φ}.) The purpose of the present paper is to show (§4) that $\omega^{\varphi}(f,.)_{p+1}$ can be replaced by $\omega^{\varphi}(f,.)_p$ and that this estimate holds for all $f \in L_p$. The corresponding estimates for approximation by reciprocals of trigonometric polynomials are derived in §3. We begin in the next section with some remarks on algebraic and trigonometric polynomial approximation.

2. **Polynomial approximation.** Error estimates for trigonometric polynomial approximation can most easily be obtained by convolution operators. For example, suppose that Λ_n is a kernel with mean value 1 on $[-\pi, \pi]$ and consider the convolution operator

$$L_n(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\Lambda_n(t)\,dt \tag{2.1}$$

It is well known and quite simple to prove that if

$$\int_{-\pi}^{\pi} |t|\,\Lambda_n(t)\,dt \le Cn^{-1}, \tag{2.2}$$

then, we have

$$\|f - L_n(f)\|_{L_p(\mathbb{T})} \le C\omega(f, n^{-1})_p \tag{2.3}$$

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for each $f \in L_p(\mathbb{T})$, $1 \leq p \leq \infty$ (with L_∞ replaced by C in the case $p = \infty$) and

$$\omega(f,t)_p := \sup_{0 \leq h \leq t} \|\Delta_h(f,.)\|_{L_p(\mathbb{T})} \tag{2.4}$$

the L_p modulus of continuity of f. Here and later, we use C to denote constants and subscripts to denote variables on which they depend (if there are any.)

If Λ_n is an even trigonometric polynomial of degree $\leq m$ then L_n is a convolution operator and $L_n(f)$ is also a trigonometric polynomial whose degree does not exceed m. The best known examples of kernels of this type are the Jackson kernels

$$k_n(t) := k_{n,r}(t) := d_n \left(\frac{\sin nt/2}{\sin t/2}\right)^{2r}, \quad r = 1, 2, \ldots \tag{2.5}$$

with the constant d_n chosen so that k_n has mean value one on $[-\pi, \pi]$: $\int_{-\pi}^{\pi} k_n(t)\,dt = 2\pi$. Then, it is easy to see (see [3]) that $d_n \approx n^{2r-1}$ and the moments of k_n satisfy

$$\int_{-\pi}^{\pi} |t|^j k_n(t)\,dt \leq C_r n^{-j}, \quad j = 0, 1, \ldots, 2r-2. \tag{2.6}$$

From (2.6), it follows that the moment condition (2.2) is valid for $r \geq 2$. We shall also use the fact that the shifted kernels $k_{n,r}(t + \delta_n)$, $\delta_n := \pi/2n$ satisfy these moment conditions (2.6) and therefore (2.3) as well.

From results on trigonometric polynomial approximation, it is possible to deduce estimates for approximation by algebraic polynomials. There are two types of estimates. The simplest of these are in terms of the ordinary modulus of continuity of $f \in L_p[-1, 1]$, $1 \leqq p \leqq \infty$. Finer results were recently given by Ditzian and Totik [1] in terms of a new modulus of continuity which has many important applications in approximation. Let $\varphi(x) = \sqrt{1 - x^2}$ and

$$\Delta_{h\varphi} f(x) = \begin{cases} f\left(x + \frac{h}{2}\varphi(x)\right) - f\left(x - \frac{h}{2}\varphi(x)\right), & x \pm \frac{h}{2}\varphi(x) \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Then, following Ditzian and Totik, we define

$$\omega^\varphi(f,t)_p := \sup_{0 < h \leqq t} \|\Delta_{h\varphi} f\|_p .$$

Ditzian and Totik have shown that for each n there is an algebraic polynomial P_n of degree $\leq n$ such that

$$\|f - P_n\|_{L_p[-1,1]} \leq C\omega^\varphi(f, \frac{1}{n})_p . \tag{2.7}$$

It will be useful to recall their method of proof of (2.7). They first establish that $\omega^\varphi(f, .)$ is equivalent to the K-functional

$$K(f,t)_p := \inf_g \{ \|f - g\|_{L_p[-1,1]} + t\|\varphi g'\|_{L_p[-1,1]} + t^2 \|g'\|_{L_p[-1,1]} \}. \tag{2.8}$$

where the infimum is taken over all absolutely continuous g. That is, they show

$$C_1\omega^{\varphi}(f,t)_p \le K(f,t)_p \le C_2\omega^{\varphi}(f,t)_p. \tag{2.9}$$

It follows from (2.9) that for each $n = 1,2,\dots$ there is a f_n such that

$$\|f-f_n\|_{L_p[-1,1]} + \frac{1}{n}\|\varphi f_n'\|_{L_p[-1,1]} + \frac{1}{n^2}\|f_n'\|_{L_p[-1,1]} \le C\omega^{\varphi}(f,\frac{1}{n})_p. \tag{2.10}$$

Moreover, their proof shows that if f is nonnegative then f_n can also be chosen to be nonnegative.

The second main point in establishing (2.7) is to approximate f_n. For this, we let $g_n(\theta) := f_n(\cos\theta)$. Then g_n is an even 2π- periodic function.

THEOREM 2.1. *If f_n and g_n are defined as above and if L_n is defined by (2.1) for some kernel Λ_n satisfying (2.2) (not necessarily a trigonometric polynomial), then*

$$\|f_n - L_n(g_n, \arccos\cdot)\|_{L_p[-1,1]} \leqq C\omega^{\varphi}(f,\frac{1}{n})_p, \quad 1\le p\le\infty.$$

This theorem is established in §7.2 of [1]. While the analysis in [1] is stated for a trigonometric kernel, the proof is exactly the same for any Λ_n.

Now to prove (2.7), it is enough to take for L_n any of the operators above with a trigonometric kernel of *degree* $\le Cn$ (for example the Jackson kernels.) This gives an even trigonometric polynomial $T_n(\theta) = L_n(g_n,\theta)$. Then $P_n(x) := T_n(\arccos x)$ satisfies (2.7).

Another property of this construction is that P_n can be used to replace f_n in (2.10):

$$\|f-P_n\|_{L_p[-1,1]} + \frac{1}{n}\|\varphi P_n'\|_{L_p[-1,1]} + \frac{1}{n^2}\|P_n'\|_{L_p[-1,1]} \le C\omega^{\varphi}(f,\frac{1}{n})_p. \tag{2.11}$$

This property follows from Theorem 7.3.1 of [1]. Although this theorem is stated for the polynomial P_n of best L_p approximation it holds with exactly the same proof for any P_n satisfying (2.7). We remark further that Theorem 7.3.1 estimates $\|\varphi P_n'\|_{L_p[-1,1]} \le Cn\omega^{\varphi}(f,\frac{1}{n})_p$ but the same proof also give $\|P_n'\|_{L_p[-1,1]} \le Cn^2\omega^{\varphi}(f,\frac{1}{n})_p$.

3. **Approximation by Reciprocals of Trigonometric Polynomials.** To prove results about approximation by reciprocals of trigonometric polynomials, we shall use the modified kernels

$$\lambda_n(t) := \frac{1}{2}\left[k_n(t-\delta_n)+k_n(t+\delta_n)\right] = c_n\left[\left(\frac{\sin\frac{n(t-\delta_n)}{2}}{\sin\frac{(t-\delta_n)}{2}}\right)^4 + \left(\frac{\sin\frac{n(t+\delta_n)}{2}}{\sin\frac{(t+\delta_n)}{2}}\right)^4\right]$$

where $\delta_n := \pi/(2n)$ and c_n is a normalizing constant chosen so that

$$\int_{-\pi}^{\pi}\lambda_n(t)\,dt = 2\pi.$$

By our earlier remarks, the kernel λ_n has the approximation properties of §2. In addition, the kernel λ_n has the following important property not held by the Jackson kernels k_n.

LEMMA 3.1. *For $s, t \in [-\pi, \pi)$ and $n \geqq 1$ we have*

$$\frac{\lambda_n(s+t)}{\lambda_n(s)} \leqq C(1+n|t|)^4. \tag{3.1}$$

PROOF. We prove (3.1) for $n \geqq 2$, the case $n = 1$ is straightforward. First we show that

$$\lambda_n(s) \leqq C \cdot c_n n^4 \qquad |s| \leqq \pi . \tag{3.2}$$

In fact, for $|s| \leqq \frac{\pi}{n}$, it follows that

$$\begin{aligned}\lambda_n(s) &\leqq c_n \left[\left(\frac{\frac{n(s-\delta_n)}{2}}{\frac{2}{\pi}\frac{(s-\delta_n)}{2}} \right)^4 + \left(\frac{\frac{n(s+\delta_n)}{2}}{\frac{2}{\pi}\frac{(s+\delta_n)}{2}} \right)^4 \right] \\ &\leqq C \cdot c_n n^4\end{aligned}$$

where we used the inequalities

$$|\sin x| \leqq \min\{1, |x|\} \qquad |x| \leqq \pi \tag{3.3}$$

$$\left|\frac{2}{\pi}x\right| \leqq |\sin x| \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{3.4}$$

For $\frac{\pi}{n} < |s| \leqq \pi$, we have by (3.4) and the monotonicity of $\sin x, |x| \leqq \frac{\pi}{2}$, that

$$\begin{aligned}\left|\sin\left(\frac{s \pm \delta_n}{2}\right)\right| &\geqq \sin\left(\frac{|s| - \delta_n}{2}\right) \\ &\geqq \frac{|s|}{\pi} - \frac{1}{2n} \\ &\geqq \frac{|s|}{2\pi}\end{aligned}$$

Hence by (3.3)

$$\begin{aligned}\lambda_n(s) &\leqq c_n \left[\frac{1}{\sin^4 \frac{(s-\delta_n)}{2}} + \frac{1}{\sin^4 \frac{(s+\delta_n)}{2}} \right] \\ &\leqq C \cdot c_n |s|^{-4} \leqq C \cdot c_n n^4\end{aligned} \tag{3.5}$$

This concludes the proof of (3.2).

We next note that

$$\lambda_n(s) \geqq C \cdot c_n n^4, \quad |s| \leqq \frac{\pi}{n} . \tag{3.6}$$

Indeed, by (3.3) and (3.4)

$$\lambda_n(s) \geqq c_n \max\left\{\left(\frac{\sin\frac{n(s-\delta_n)}{2}}{\sin\frac{(s-\delta_n)}{2}}\right)^4, \left(\frac{\sin\frac{n(s+\delta_n)}{2}}{\sin\frac{(s+\delta_n)}{2}}\right)^4\right\}$$
$$\geqq C\,c_n \min\left\{\left(\frac{\frac{n(s-\delta_n)}{\pi}}{\frac{(s-\delta_n)}{2}}\right)^4, \left(\frac{\frac{n(s+\delta_n)}{\pi}}{\frac{(s+\delta_n)}{2}}\right)^4\right\} \geqq C\cdot c_n n^4.$$

By (3.2) and (3.6) we have for $|s| \leqq \frac{\pi}{n}$ and all $t \in \mathrm{T}$

$$\frac{\lambda_n(s+t)}{\lambda_n(s)} \leqq C.$$

Thus (3.1) follows for $|s| \leqq \frac{\pi}{n}$.

Consider now the case $\frac{\pi}{n} < |s|$. Since

$$\sin^2\frac{n(s-\delta_n)}{2} + \sin^2\frac{n(s+\delta_n)}{2} = 1$$

it follows that

$$\max\{\sin^4\frac{n(s-\delta_n)}{2}, \sin^4\frac{n(s+\delta_n)}{2}\} \geqq \frac{1}{4}.$$

Therefore, we have

$$\lambda_n(s) \geqq \frac{1}{4}c_n \min\left\{\frac{1}{\sin^4\frac{(s-\delta_n)}{2}}, \frac{1}{\sin^4\frac{(s+\delta_n)}{2}}\right\} \tag{3.7}$$
$$\geqq C\cdot c_n|s|^{-4},$$

where again we used (3.3) for the last inequality. Combining (3.2) and (3.7) we obtain

$$\frac{\lambda_n(s+t)}{\lambda_n(s)} \leqq C(n|s|)^4 \leqq Cn^4 \tag{3.8}$$

which yields (3.1) for $|t| \geqq \frac{\pi}{4}$.

We now consider the remaining case $|t| < \frac{\pi}{4}$ and $\frac{\pi}{n} < |s|$.

If $|s+t| \leqq \frac{\pi}{n}$, then

$$|s| \leqq |t| + \frac{\pi}{n}$$

and it follows by the left inequality in (3.8) that

$$\frac{\lambda_n(s+t)}{\lambda_n(s)} \leqq C(n|s|)^4 \leqq C(1+n|t|)^4.$$

If $\frac{\pi}{n} < |s+t| \leqq \pi$ then by (3.5) and (3.7)

$$\frac{\lambda_n(s+t)}{\lambda_n(s)} \leqq C\left(\frac{|s|}{|s+t|}\right)^4$$
$$\leqq C\left(1+\frac{|t|}{|s+t|}\right)^4$$
$$\leqq C\left(1+n|t|\right)^4.$$

Finally if $\pi < |s+t| < \pi + \frac{\pi}{4}$ (since $|t| < \frac{\pi}{4}$), then due to the periodicity of $\lambda_n(s)$ we have by (3.5)

$$\begin{aligned}\lambda_n(s+t) &= \lambda_n(2\pi - |s+t|)\\ &\leqq Cc_n\big|2\pi - |s+t|\big|^{-4}\\ &\leqq Cc_n\end{aligned}$$

and combining with (3.7), we obtain

$$\frac{\lambda_n(s+t)}{\lambda_n(s)} \leqq C\,.$$

Thus the proof of (3.1) is complete. ∎

We are going to use the kernel λ_n to average the function f to be approximated. Let $f \in L_p(\mathbf{T})$, $1 \leqq p < \infty$, be nonnegative and assume $f \not\equiv 0$. Define the averaged function

$$\tilde{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+s)\lambda_n(s)\,ds\,.$$

Then we have

LEMMA 3.2. *If $f \in L_p(\mathbf{T})$, $1 \leqq p \leqq \infty$, and $\tilde{f}$ are as above, then*

$$\begin{aligned}&(i)\quad \|f - \tilde{f}\|_{L_p(\mathbf{T})} \leqq C\omega(f, \frac{1}{n})_p,\\ (3.9)\qquad &(ii)\quad \omega(\tilde{f}, t)_p \leqq C\omega(f,t)_p,\\ &(iii)\quad \sup_{-\pi\leqq x<\pi} \frac{\tilde{f}(x)}{\tilde{f}(x+t)} \leqq C(1+n|t|)^4, \qquad |t| \leqq \pi.\end{aligned}$$

PROOF. Statement (i) follows from (2.3). Since λ_n is positive and has mean value one, (ii) follows immediately from the identity $\Delta_h\tilde{f} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_h(f, x+t)\lambda_n(t)\,dt$. To prove (iii), we notice that since $f \not\equiv 0$ and nonnegative, it follows that $\tilde{f} > 0$ on $\mathbf{T}$ so that the quotient is well defined. By (3.1)

$$\lambda_n(s) \leqq C(1+n|t|)^4\lambda_n(s-t)\,.$$

Thus

$$\begin{aligned}\tilde{f}(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+s)\lambda_n(s)ds\\ &\leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+s)C(1+n|t|)^4\lambda_n(s-t)ds\\ &\leqq C(1+n|t|)^4\int_{-\pi}^{\pi} f(x+s)\lambda_n(s-t)ds\\ &= C(1+n|t|)^4\tilde{f}(x+t)\end{aligned}$$

and (iii) is proved. ∎

We are ready to prove the main result of this section.

THEOREM 3.3. *Let $f \in L_p(\mathbb{T})$, $1 \leqq p \leq \infty$, be nonnegative and assume $f \not\equiv 0$. Then for each $n \geqq 1$ there exists a trigonometric polynomial T_n of degree $\leqq n$ such that*

$$\left\| f - \frac{1}{T_n} \right\|_p \leqq C\omega(f, \frac{1}{n})_p .$$

PROOF. If $f \equiv c$, $c \neq 0$, then we can take $T_n := \frac{1}{c}$. Therefore we can assume that f is not a constant. Let

$$k_n(t) = d_n \left(\frac{\sin nt/2}{\sin t/2} \right)^8$$

where d_n is the normalizing constant such that

$$\int_{-\pi}^{\pi} k_n(t)\, dt = 2\pi .$$

We shall use the convolution operator $L_n(g) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x+t) k_n(t)\, dt$. By (2.6),

$$\int_{-\pi}^{\pi} (1 + n|t|)^6 k_n(t)\, dt \leqq C .$$

We define $f_\epsilon(x) = f(x) + \epsilon$, with $\epsilon > 0$ to be chosen later and let $g(x) = \tilde{f}_\epsilon(x)$ be the averaged function. The function

$$T_n(x) := L_n(\frac{1}{g}, x)$$

is well defined since $g \geqq \epsilon$ and is a trigonometric polynomial of degree $\leq 3n$. It follows from the positivity of the operator L_n that (see [2]) $L_n(g) L_n(1/g) \geq L_n(1) = 1$ and therefore

$$\frac{1}{T_n(x)} \leqq L_n(g) . \tag{3.10}$$

As in [2], we consider two sets,

$$E_1 = \left\{ x \in [-\pi, \pi) : \frac{1}{T_n(x)} > g(x) \right\} \quad \text{and} \quad E_2 = [-\pi, \pi) \setminus E_1 .$$

Then by (3.10)

$$\left\| \frac{1}{T_n(x)} - g(x) \right\|_{L_p(E_1)} \leqq \| g - L_n(g) \|_{L_p(\mathbb{T})} \leqq C\omega(g, \frac{1}{n})_p \tag{3.11}$$

where for the last inequality we used (2.3).

For $x \in E_2$, we have

$$\begin{aligned}
0 \leqq g(x) - \frac{1}{T_n(x)} &= \frac{g(x)}{T_n(x)}\left[T_n(x) - \frac{1}{g(x)}\right] \\
&\leqq g^2(x)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{g(x+t)} - \frac{1}{g(x)}\right]k_n(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}[g(x) - g(x+t)]\frac{g(x)}{g(x+t)}k_n(t)\,dt \\
&\leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g(x) - g(x+t)\right|C(1+n|t|)^4k_n(t)\,dt
\end{aligned}$$

where for the last inequality we used the property (iii) of Lemma 3.2. The kernel $\Lambda_n(t) := (1+n|t|)^4k_n(t)$ satisfies (2.3) and therefore

$$\begin{aligned}
\left\|g(x) - \frac{1}{T_n(x)}\right\|_{L_p(E_2)} &\leq C\left\|\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(x+t) - g(x)|(1+n|t|)^4K_n(t)\,dt\right\|_{L_p(\mathbb{T})} \\
&\leq C\,\omega(g, \frac{1}{n})_p
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) with Lemma 3.2(i) and (ii) yields

$$\begin{aligned}
\|f - \frac{1}{T_n}\|_{L_p(\mathbb{T})} &\leqq \|f - f_\epsilon\|_{L_p(\mathbb{T})} + \|f_\epsilon - \tilde{f}_\epsilon\|_{L_p(\mathbb{T})} + \left\|g - \frac{1}{T_n}\right\|_{L_p(\mathbb{T})} \\
&\leqq C\left(\epsilon + \omega(f, \frac{1}{n})_p\right).
\end{aligned}$$

Thus the choice $\epsilon = \omega\left(f, \frac{1}{n}\right)_p$ (which is positive since f is not a constant) concludes the proof of Theorem 3.3. ∎

4. **Approximation by Reciprocals of Algebraic Polynomials.** We prove in this section the following improvement of the result of Leviatan, Levin and Saff [2].

THEOREM 4.1. *Let $f \in L_p[-1,1]$, $1 \leqq p \leqq \infty$, be nonnegative and assume $f \not\equiv 0$. Then for each $n \geqq 1$ there exists an algebraic polynomial P_n of degree $\leqq n$ such that*

$$\left\|f - \frac{1}{P_n}\right\|_p \leqq C\omega^\varphi(f, \frac{1}{n})_p. \tag{4.1}$$

Let f_n be the nonnegative function which satisfies (2.10). We shall follow the construction of the previous section by setting $g_n(\theta) := f_n(\cos\theta)$ and letting $\tilde{g}_n(\theta)$ be the averaged function of g_n (using the same n for averaging). Note that $g_n \in L_p(\mathbb{T})$. Now writing $\tilde{f}_n(x) = \tilde{g}_n(\theta)$ where $x = \cos\theta$ we observe that by virtue of the evenness of $\lambda_n(t)$, $\tilde{f}_n(x)$ is an algebraic polynomial of degree $\leqq 2n$. We summarize the properties of $\tilde{f}_n$ in the following:

LEMMA 4.2. *With f_n and $\tilde{f}_n$ defined as above, we have*

$$\|f_n - \tilde{f}_n\|_p \leqq C\omega^\varphi(f, \frac{1}{n})_p \tag{4.2}$$

$$\|\varphi \tilde{f}_n'\|_p \leqq Cn\omega^\varphi(f, \frac{1}{n})_p \tag{4.3}$$

$$\|\tilde{f}_n''\|_p \leqq Cn^2\omega^\varphi(f, \frac{1}{n})_p . \tag{4.4}$$

PROOF. Since $\lambda_n(t)$ is a kernel satisfying the properties of § 2, $\tilde{f}_n$ is one of the polynomials P_n which satisfy (2.11). ■

We are ready now to prove Theorem 4.1.

Proof of Theorem 4.1. Again, we may assume that $f \not\equiv c$ We follow the ideas of the proof of Theorem 3.3. We let $g_\epsilon(\theta) = \tilde{g}_n(\theta) + \epsilon$ where $\tilde{g}_n(\theta) = \tilde{f}_n(\cos\theta)$ and let

$$P_n(x) := L_n(\frac{1}{g_\epsilon}, \arccos x) .$$

Again we define

$$E_1 = \left\{x \in [-1, 1] : \frac{1}{P_n(x)} > g_\epsilon(\arccos x)\right\}$$

and $E_2 = [-1, 1] \setminus E_1$. Then, we have

$$\begin{aligned} &\left\|\frac{1}{P_n(x)} - g_\epsilon(\arccos x)\right\|_{L_p(E_1)} \\ &\leqq \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[g_\epsilon(\arccos x + t) - g_\epsilon(\arccos x)\right]k_n(t)dt\right\|_{L_p[-1,1]} . \end{aligned} \tag{4.5}$$

Notice that the L_p norm is with respect to x. Similarly,

$$\begin{aligned} &\left\|\frac{1}{P_n(x)} - g_\epsilon(\arccos x)\right\|_{L_p(E_2)} \\ &\leqq C\left\|\int_{-\pi}^{\pi}\left|g_\epsilon(\arccos x + t) - g_\epsilon(\arccos x)\right|(1 + n|t|)^4 k_n(t)\,dt\right\|_{L_p[-1,1]} . \end{aligned} \tag{4.6}$$

In other words, we have

$$\begin{aligned} &\left\|\tilde{f}_n + \epsilon - \frac{1}{P_n}\right\|_{L_p[-1,1]} \\ &\leqq C\left\|\int_{-\pi}^{\pi}\left|g_\epsilon(\arccos x + t) - g_\epsilon(\arccos x)\right|(1 + n|t|)^4 k_n(t)\,dt\right\|_{L_p[-1,1]} . \end{aligned}$$

Now the kernel $\Lambda_n(t) := (1 + n|t|)^4 k_n(t)$ satisfies

$$\int_{-\pi}^{\pi}|t|\Lambda_n(t)\,dt \leq C/n$$

and therefore by Theorem 2.1, we have

$$\left\|\tilde{f}_n + \epsilon - \frac{1}{P_n}\right\|_p \leqq C\omega^{\varphi}(f, \frac{1}{n})_p . \tag{4.7}$$

Finally choosing $\epsilon = \omega^{\varphi}\left(f, \frac{1}{n}\right)_p$ we get by (4.2) and (4.7),

$$\begin{aligned}\left\|f - \frac{1}{P_n}\right\|_{L_p[-1,1]} &\leqq \|f - \tilde{f}_n\|_{L_p[-1,1]} + \epsilon + \left\|\tilde{f}_n + \epsilon - \frac{1}{P_n}\right\|_{L_p[-1,1]}\\ &\leqq C\omega^{\varphi}\left(f, \frac{1}{n}\right)_p\end{aligned}$$

and our proof is complete. ∎

REFERENCES

1. Z. Ditzian and V. Totik *Moduli of smoothness*, Springer-Verlag, Berlin 1987.

2. D. Leviatan, A. L. Levin and E. B. Saff *On approximation in the L_p-norm by reciprocals of polynomials*, J. Approx. Theory **57** 1989 322–331.

3. G. G. Lorentz *Approximation of functions*, Holt Rinehart and Winston, New York, 1966.

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