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CONSTRUCTIVE APPROXIMATION

Polynomial Approximation in L_p $(0 < p < 1)$

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Abstract. We prove that for $f \in L_p$, $0 < p < 1$, and k a positive integer, there exists an algebraic polynomial P_n of degree $\le n$ such that

$$\|f - P_n\|_p \le C\omega_k^\varphi\left(f, \frac{1}{n}\right)_p,$$

where $\omega_k^\varphi(f, t)_p$ is the Ditzian–Totik modulus of smoothness of f in L_p, and C is a constant depending only on k and p. Moreover, if f is nondecreasing and $k \le 2$, then the polynomial P_n can also be taken to be nondecreasing.

1. Introduction

We are interested in the approximation of functions $f \in L_p(I)$, $0 < p < 1$, $I = [-1, 1]$, by algebraic polynomials. Such approximation has previously been studied by other authors, most notably, Storozhenko, Krotov, and Oswald [S–K–O] and Khodak [K]. Our main departure from these previous works is that we shall prove direct estimates for the error of polynomial approximation in terms of the Ditzian–Totik modulus of smoothness. This modulus measures smoothness differently at the endpoints of I than in the interior. Such dependence on the position of the point is crucial if we wish to characterize functions with a certain error of polynomial approximation (see [D–T]). We, however, do not in this paper discuss inverse estimates in terms of this modulus.

A second variant of our work is to consider the approximation of monotone functions by monotone algebraic polynomials in L_p, $0 < p < 1$. We establish the same estimates as for the unconstrained case but only for the first- and second-order moduli. There is a result of Shvedov [S] which says that such estimates cannot hold for smoothness order greater than 2.

The usual estimates for approximating $f \in L_p(I)$ by algebraic polynomials are described in terms of the ordinary kth order modulus of smoothness of f. If J is

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an interval, we let

$$\omega_k(f, t, J)_p := \sup_{0<h\le t}\left(\int_J |\Delta_h^k(f, x, J)|^p\, dx\right)^{1/p}.$$

Here, Δ_h^k is the *symmetric* difference:

$$\text{(1.1)}\qquad \Delta_h^k(f, x, J) := \begin{cases} \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} f\left(x - \frac{k}{2}h + ih\right) & \text{if } x \pm \frac{k}{2}h \in J, \\ 0 & \text{otherwise.}\end{cases}$$

We reserve the notation I for the interval $[-1, 1]$ throughout this paper. In the case of this interval, we simply write $\omega_k(f, t)_p := \omega_k(f, t, I)_p$. A fundamental inequality (usually called a Jackson inequality) for algebraic polynomial approximation of $f \in L_p(I)$, $0 < p < \infty$, says that for each positive integer k there exists an algebraic polynomial P_n of degree $\le n$ such that

$$\text{(1.2)}\qquad \|f - P_n\|_p \le C\omega_k\left(f, \frac{1}{n}\right)_p,$$

where C is a constant depending only on k and p (see [S–K–O]).

We shall improve upon (1.2) by incorporating the position of the point $x \in I$ into the analysis. If $\varphi(x) := \sqrt{1 - x^2}$, the Ditzian–Totik modulus of smoothness of $f \in L_p(I)$ is defined by

$$\omega_k^\varphi(f, t)_p := \sup_{0<h\le t}\left(\int_{-1}^{1} |\Delta_{h\varphi(x)}^k(f, x, I)|^p\, dx\right)^{1/p},$$

where

$$\text{(1.3)}\quad \Delta_{h\varphi(x)}^k(f, x, I) := \begin{cases} \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} f\left(x - \frac{k}{2}h\varphi(x) + ih\varphi(x)\right) & x \pm \frac{k}{2}h\varphi(x) \in I, \\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that $\omega_k^\varphi(f, \cdot)_p$ is defined for each $f \in L_p(I)$ and satisfies $\omega_k^\varphi(f, t)_p \le C\|f\|_p(I)$ for sufficiently small $t > 0$ (see (D–T, p. 21]).

For $1 \le p \le \infty$, Ditzian and Totik improved the estimate (1.2) by showing that $\omega_k(f, \cdot)_p$ can be replaced by the smaller $\omega_k^\varphi(f, \cdot)_p$. They proved that for $f \in L_p(I)$, $1 \le p \le \infty$, and positive integer k, there exists an algebraic polynomial $P_n(x)$ of degree $\le n$ such that

$$\|f - P_n\|_p \le C\omega_k^\varphi\left(f, \frac{1}{n}\right)_p,$$

where C depends only on k.

In this paper, we shall show that this conclusion is also valid for $0 < p < 1$:

Theorem 1.1. *Let $f \in L_p(I)$, $0 < p < 1$, and k be a positive integer. Then, for each $n \ge N$ (with N a constant depending only on p and k), there exists an algebraic* *polynomial P_n of degree $\leq n$ such that*

$$\|f - P_n\|_p \leq C\omega_k^\varphi\left(f, \frac{1}{n}\right)_p, \tag{1.4}$$

where C depends only on k and p. Moreover, if f is nondecreasing on I, then for $k \leq 2$ the polynomial P_n in (1.4) can also be taken to be nondecreasing.

Remark. (i) For $1 \leq p < \infty$ similar estimates for monotone functions are due to Leviatan and Yu [L–Y] (see also [Y]).

(ii) By the aforementioned result of Shvedov [S], (1.4) cannot hold for $k \geq 3$ for monotone approximation.

We devote Sections 4 and 5 to the proof of Theorem 1. In Section 4, we deal with the case of nonconstrained approximation. These estimates are later applied in Section 5 in order to settle the question of monotone approximation. However, the monotone case is much more involved and we start in Section 2 with a construction of continuous monotone piecewise linear functions, namely, splines of order 2, which yields "good" approximation to a monotone $f \in L_p(I)$. This result is interesting in its own right and plays a critical role in the construction of monotone polynomial approximants. We collect in Section 3 some well-known results about polynomials that are needed in the proof of Theorem 1.1. Throughout the paper $0 < p < 1$ is fixed.

2. Monotone Piecewise Linear Approximants

Let $a = \xi_0 < \xi_1 < \cdots < \xi_n = b$ be such that adjacent $I_j = [\xi_{j-1}, \xi_j]$, $j = 1, \ldots, n$, have comparable lengths, i.e.,

$$\frac{|I_{j\pm 1}|}{|I_j|} \leq C_0 \tag{2.1}$$

with C_0 an absolute constant. We shall be interested in this section in the approximation of $f \in L_p[a, b]$ by the elements of $\mathscr{S}$, where $\mathscr{S}$ denotes the class of piecewise linear functions on $[a, b]$ for this partition. Each function $S \in \mathscr{S}$ is completely determined by its left- and right-hand values $S(\xi_j \pm)$, $j = 1, \ldots, n-1$, and the values $S(\xi_0)$, $S(\xi_n)$.

If f is a function in $L_p(J)$, $0 < p \leq 1$, J an interval, then a polynomial P of degree k is a *near best* L_p approximation (with constant M) to f from among all polynomials of degree $\leq k$ if

$$\|f - P\|_p(J) \leq ME_k(f, J)_p, \tag{2.2}$$

where $E_k(f, J)_p$ is the error of best approximation to f on J in the L_p (quasi-)norm from among all polynomials of degree $\leq k$. Of course, if $M = 1$, then P is a best approximant.

We shall frequently make use of the following remark which was proved in [D–P]. If P is a near-best approximation to f with constant M on an interval J, then for on any larger interval $\tilde{J}$ we have

$$\|f - P\|_p(\tilde{J}) \le CME_k(f, \tilde{J})_p, \tag{2.3}$$

with a constant that depends only on p, k, and the ratio $|\tilde{J}|/|J|$. That is, P is a near-best approximant on the larger interval $\tilde{J}$ with constant CM.

Given *any measurable function* f and an interval J, we can speak of a best $L_1(J)$ approximant l to f from linear functions (i.e., polynomials of degree ≤ 1) in the following sense. There should exist a measurable function h with $|h| = 1$ on J such that $h(x) = \operatorname{sgn}(f - l)(x)$, whenever $x \in J$ and $|f(x) - l(x)| > 0$ and h is orthogonal to all linear functions on J. Brown and Lucier [B–L] have shown that for each $f \in L_p(J)$, $0 < p \le 1$, there exists such linear functions l, and moreover, l is a near-best L_p approximant on J.

The orthogonality condition implies that when $f \ne l$, a.e., then

$$\operatorname{meas}\{x \in J\colon f(x) - l(x) > 0\} = \operatorname{meas}\{x \in J\colon f(x) - l(x) < 0\}. \tag{2.4}$$

In particular, in this case, $f - l$ has a (weak) sign change on J.

If we suppose in addition that f is nondecreasing on J, then it is easy to see that any l (which is a best L_1 approximation to f in the above sense) must also be nondecreasing on J. Indeed, if $f = l$ on a set of positive measure this is obvious. On the other hand, if $f \ne l$ a.e., then $h := \operatorname{sgn}(f - l)$ must have at least two sign changes (because it is orthogonal to all linear functions), say at the points ξ and η with $\xi < \eta$. Then $l(\xi) \le l(\eta)$ and therefore l is nondecreasing.

Now we return to the question of piecewise polynomial approximation. For each of our intervals I_j we let l_j be a best L_1 approximant to f on I_j in the sense given above. Then from our above remarks, l_j is nondecreasing and from (2.3) we have that for any interval $\tilde{I}_j \supseteq I_j$ of comparable length to $|I_j|$

$$\|f - l_j\|_p(\tilde{I}_j) \le ME_1(f, \tilde{I}_j)_p \tag{2.5}$$

with M depending only on p and on the ratio of the lengths of $\tilde{I}_j$ and I_j.

We define the piecewise linear function $S \in \mathcal{S}$ by

$$S(x) := l_j(x), \qquad \xi_{j-1} < x < \xi_j, \tag{2.6}$$

and prove the following.

Theorem 2.1. *For any interval $[a, b]$ and partition $a = \xi_0 < \xi_1 < \cdots < \xi_n = b$ there is a piecewise linear function $\bar{S} \in \mathcal{S}$ with the following properties*:

- (i) *$\bar{S}$ is nondecreasing*;
- (ii) *there is a constant $M_0 > 0$ depending only on p and the constant C_0 of* (2.1) *such that for $j = 1, \ldots, n$, $\bar{l}_j$ satisfies* (2.5) *for an interval $\tilde{I}_j$ with $I_j \subset \tilde{I}_j \subset \bar{I}_j := I_{j-2} \cup I_{j-1} \cup I_j \cup I_{j+1} \cup I_{j+2}$ (for the purposes of this formula, $I_k := \varnothing$, $k \le 0$, and $k > n$).*
- (iii) *$\bar{S}(\xi_0+) \ge S(\xi_0+)$ and $\bar{S}(\xi_n-) \le S(\xi_n-)$.*

Proof. The proof is by induction on the number of intervals n. Let M_1 be a constant such that (2.5) is valid for all $j = 1, \ldots, n$ and any $\tilde{I}_j$ with $I_j \subset \tilde{I}_j \subset \bar{I}_j$.

We have observed earlier that M_1 depends only on p and C_0. We shall show that the theorem is valid for $M_0 := C_1 M_1$, with $C_1 \geq 1$ a sufficiently large but fixed constant depending only on p and the constant C_0 of (2.1). If $n = 1$, we can take $\bar{S} = l_1 = S$.

Assume then that the theorem is true for any number of intervals $< n$. We let $A_j := \{y: y = l_j(x),\ x \in I_j\} = [\alpha_j, \beta_j],\ j = 1, \ldots, n$, be the range of l_j on I_j. If $f \neq l_i$ a.e. on I_i, $i = j, j+1$, then by (2.4), $f - l_i$ changes sign at some point $\eta_i \in I_i^0$ (the interior of I_i), $i = j, j+1$. Since

$$l_j(\eta_j) \leq f(\eta_j +) \leq f(\eta_{j+1} -) \leq l_{j+1}(\eta_{j+1}),$$

we have $\alpha_j \leq \beta_{j+1}$. Clearly this also holds if $f = l_i$ on a set of positive measure in I_i for one or both of the values $i = j, j+1$.

We define m as the largest integer such that $\bigcap_{i=1}^m A_i \neq \varnothing$, and we consider two possibilities. If $m = 1$ we have $\beta_1 < \alpha_2$ and we define $\bar{S} = l_1$ on I_1. By our induction hypothesis there is a function $\bar{S}$ for $[\xi_1, \xi_n]$. The composite function $\bar{S}$ is nondecreasing by virtue of (iii):

$$\bar{S}(\xi_1 -) = \beta_1 < \alpha_2 = S(\xi_1 +) \leq \bar{S}(\xi_1 +).$$

Properties (ii) and (iii) also follow from our induction hypothesis and the definition of $\bar{S}$.

The second possibility is $m \geq 2$. Let $\alpha \in A_1 \cap \cdots \cap A_m$. We first define $\bar{S}(\xi_j \pm) := \alpha, j = 1, \ldots, m-1$, $\bar{S}(\xi_m -) := \alpha$. We have yet to define $\bar{S}(\xi_0)$ which will then completely define $\bar{S}$ on $[\xi_0, \xi_m)$. To define $\bar{S}(\xi_0)$ we consider two cases.

Case 1. $l_2(\xi_0) \leq l_1(\xi_0) = S(\xi_0)$.
In this case we define $\bar{S}(\xi_0) = S(\xi_0) = l_1(\xi_0)$. Then on $[\xi_0, \xi_1]$ we have $l_2(x) \leq \bar{S}(x) \leq l_1(x)$ where we used that $\bar{S}(\xi_1) = \alpha \geq l_2(\xi_1)$. Hence, with $\tilde{I}_1 = I_1 \cup I_2$, we have

$$\begin{aligned}\|l_1 - \bar{l}_1\|_p(I_1) &\leq \|l_1 - l_2\|_p(I_1)\\ &\leq C[\|f - l_1\|_p(I_1) + \|f - l_2\|_p(I_1)]\\ &\leq C[\|f - l_1\|_p(\tilde{I}_1) + \|f - l_2\|_p(\tilde{I}_1)]\\ &\leq C M_1 E_1(f, \tilde{I}_1)_p,\end{aligned}$$

where the last inequality uses (2.5). We can now replace I_1 by $\tilde{I}_1$ on the left of our inequality (because the norm of l on two intervals of comparable size are equivalent). Therefore,

$$\|f - \bar{l}_1\|_p(\tilde{I}_1) \leq C[\|f - l_1\|_p(\tilde{I}_1) + \|l_1 - \bar{l}_1\|_p(\tilde{I}_1)]$$

and (ii) follows for $j = 1$ and $M_0 := C_1 M_1$ provided C_1 is sufficiently large but depending only on p and C_0.

Case 2. $l_2(\xi_0) > l_1(\xi_0) = S(\xi_0)$.
Then l_1 and l_2 intersect at some point ξ with $\xi_0 < \xi \leq \xi_1$. If $\xi < \xi_1$ we define $\bar{l}_1$ as the linear function which takes the values α at ξ_1 and the common value $l_1(\xi) = l_2(\xi)$ at ξ. If $\xi = \xi_1$ we can take $\bar{l}_1 := l_1$. Then $l_1(x) \leq \bar{l}_1(x) \leq l_2(x)$, $\xi_0 \leq$

$x \le \xi$, $l_2(x) \le \bar{l}_1(x) \le l_1(x)$, $\xi \le x \le \xi_1$. Hence again we have

$$\|\bar{l}_1 - l_1\|_p(I_1) \le \|l_1 - l_2\|_p(I_1)$$

and arguing as in Case 1 we have property (ii) for $\bar{l}_1$. This completes the definition of $\bar{S}$ on $[\xi_0, \xi_1]$.

We check next that property (ii) is valid for our definition of $\bar{S}$ on I_j, $j = 2, \ldots, m-1$. On such an interval, $\bar{S}(x) \equiv \alpha$. Let $\eta_j \in I_j$ be the point where $l_j(\eta_j) = \alpha$. Then for $\eta_j \le x \le \xi_j$ we have $l_{j+1}(x) \le \alpha = \bar{l}_j(x) \le l_j(x)$. Hence, for $J_+ := [\eta_j, \xi_j]$, we have

$$\begin{aligned}\|l_j - \alpha\|_p(J_+) &\le \|l_j - l_{j+1}\|_p(J_+)\\ &\le C[\|f - l_j\|_p(J_+) + \|f - l_{j+1}\|_p(J_+)].\end{aligned}$$

Similarly for $J_- := [\xi_{j-1}, \eta_j]$, we have

$$\|l_j - \alpha\|_p(J_-) \le C[\|f - l_{j-1}\|_p(J_-) + \|f - l_j\|_p(J_-)].$$

Hence

$$\|l_j - \bar{l}_j\|_p(I_j) \le C[\|f - l_{j-1}\|_p(I_j) + \|f - l_j\|_p(I_j) + \|f - l_{j+1}\|_p(I_j)].$$

We can replace I_j by $\tilde{I}_j := I_{j-1} \cup I_j \cup I_{j+1}$ on the left and right and obtain as before

$$\begin{aligned}\|f - \bar{l}_j\|_p(\tilde{I}_j) &\le C[\|f - l_j\|_p(\tilde{I}_j) + \|l_j - \bar{l}_j\|_p(\tilde{I}_j)]\\ &\le C[\|f - l_{j-1}\|_p(\tilde{I}_j) + \|f - l_j\|_p(\tilde{I}_j) + \|f - l_{j+1}\|_p(\tilde{I}_j)]\\ &\le CM_1 E_1(f, \tilde{I}_j)_p,\end{aligned}$$

since each of the l_{j-1}, l_j, and l_{j+1} are near best on $\tilde{I}_j$. This shows that (ii) holds for $\bar{l}_j$. Since $\bar{l}_m \equiv \bar{l}_{m-1} \equiv \alpha$, property (ii) holds for $j = m$ if we define $\tilde{I}_m := \tilde{I}_{m-1}$.

In summary, we have defined a monotone piecewise linear function $\bar{S}$ on $[\xi_0, \xi_m]$ and $\bar{S}$ has property (ii) and $\bar{S}(\xi_0+) \ge S(\xi_0+)$. By our induction hypothesis there is an $\bar{S}$ on $[\xi_m, \xi_n]$ which also satisfies (i)–(iii) for that interval. By the definition of m we must have $\alpha \le S(\xi_m+) \le \bar{S}(\xi_m+)$. Hence the composite function $\bar{S}$ is monotone nondecreasing and satisfies (iii). Property (ii) also follows from the induction hypothesis. ■

We now wish to alter the definition of $\bar{S}$ to make it continuous. We first note that in Theorem 2.1 the intervals $\tilde{I}_j$ can be taken as $\bar{I}_j$ by enlarging the constant M_0 if necessary.

Theorem 2.2. *Under the hypothesis of Theorem* 2.1, *there is a nondecreasing piecewise linear function $S^* \in \mathcal{S}$ satisfying* (i) *and* (ii) *of Theorem* 2.1 *with $\bar{I}_j$ replaced by $I_j^* := \bigcup_{j-3}^{j+3} I_\nu$ and the additional property that S^* is continuous.*

Proof. We let $S^*(\xi_0) = \bar{S}(\xi_0)$ and $S^*(\xi_n) = \bar{S}(\xi_n)$. Then, for $1 \le k < n$, we set

$$S^*(\xi_k) = \begin{cases}\bar{S}(\xi_k-) & \text{if slope}\,\bar{l}_k \le \text{slope}\,\bar{l}_{k+1},\\ \bar{S}(\xi_k+) & \text{if slope}\,\bar{l}_k > \text{slope}\,\bar{l}_{k+1},\end{cases}$$

and define S^* on I_k to be the linear function l_k^* connecting $S^*(\xi_{k-1})$ and $S^*(\xi_k)$.

Evidently S^* is continuous and increasing. Therefore, we have only to show that l_k^* is a near-best L_p approximant for I_k^*. First assume $1 < k < n$. We consider four cases.

Case 1. $S^*(\xi_{k-1}) = \bar{S}(\xi_{k-1}-)$ and $S^*(\xi_k) = \bar{S}(\xi_k-)$.
In this case slope $\bar{l}_{k-1} \le$ slope $\bar{l}_k \le$ slope $\bar{l}_{k+1}$, thus $\bar{l}_{k-1}(x) \le l_k^*(x) \le \bar{l}_k(x)$, $x \in [\xi_{k-1}, \xi_k]$. As in the proof of Theorem 2.1, we obtain

$$\|l_k^* - \bar{l}_k\|(I_k) \le C[\|f - \bar{l}_{k-1}\|_p(I_k) + \|f - \bar{l}_k\|_p(I_k)].$$

We can replace I_k by I_k^* on the left and obtain as before

$$\begin{aligned}\|f - l_k^*\|_p(I_k^*) &\le C[\|f - \bar{l}_k\|_p(I_k^*) + \|l_k^* - \bar{l}_k\|_p(I_k^*)] \\ &\le C[\|f - \bar{l}_k\|_p(I_k^*) + \|f - \bar{l}_k\|_p(I_k) + \|f - \bar{l}_{k-1}\|_p(I_k)] \\ &\le CM_0[E_1(f, I_k^*)_p + E_1(f, \bar{I}_{k-1})_p] \le CM_0 E_1(f, I_k^*)_p.\end{aligned}$$

Case 2. $S^*(\xi_{k-1}) = \bar{S}(\xi_{k-1}-)$ and $S^*(\xi_k) = \bar{S}(\xi_k+)$.
In this case slope $\bar{l}_{k-1} \le$ slope $\bar{l}_k$ and slope $\bar{l}_{k+1} <$ slope $\bar{l}_k$, thus $\bar{l}_{k-1}(x) \le l_k^*(x) \le \bar{l}_{k+1}(x)$, $x \in [\xi_{k-1}, \xi_k]$, and again we get as before that l_k^* is a near-best L_p approximant for I_k^*.

Case 3. $S^*(\xi_{k-1}) = \bar{S}(\xi_{k-1}+)$ and $S^*(\xi_k) = \bar{S}(\xi_k-)$.
In this case, $l_k^*(x) = \bar{l}_k(x)$, $x \in [\xi_{k-1}, \xi_k]$, so obviously l_k^* is a near-best L_p approximant for I_k^*.

Case 4. $S^*(\xi_{k-1}) = \bar{S}(\xi_{k-1}+)$ and $S^*(\xi_k) = \bar{S}(\xi_k+)$.
In this case slope $l_{k+1} <$ slope $\bar{l}_k <$ slope $\bar{l}_{k-1}$, thus $\bar{l}_k(x) \le l_k^*(x) \le \bar{l}_{k+1}(x)$, $x \in [\xi_{k-1}, \xi_k]$, so that as in Case 1 we get that l_k^* is a near-best L_p approximant for I_k^*.

If $k = 1$, we have two cases. The first is $S^*(\xi_1) := \bar{S}(\xi_1-)$ and $l_1^* := \bar{l}_1$ so there is nothing new to prove. In the second case, $S^*(\xi_1) := \bar{S}(\xi_1+)$ and slope $\bar{l}_2 <$ slope $\bar{l}_1$. Therefore, $\bar{l}_1(x) \le l_1^*(x) \le \bar{l}_2(x)$, $x \in I_1$, and the proof is completed as before. The remaining $k = n$ is dealt with in the same way. This completes the proof of (ii) and the theorem. ■

For our next result and for later use, we introduce the following averaged modulus of smoothness on an interval J:

$$(2.7) \qquad w_k(f, t, J)_p := \left(t^{-1} \int_0^t \int_J |\Delta_s^k(f, x, J)|^p \, dx \, ds\right)^{1/p}.$$

Then w_k is equivalent to ω_k in the sense that (see [P–P] or [D–L])

$$(2.8) \qquad C^{-1} w_k(f, t, J)_p \le \omega_k(f, t, J)_p \le C w_k(f, t, J)_p, \qquad t > 0,$$

with the constant $C \ge 1$ depending only on k and p.

We also recall Whitney's theorem which is known to hold for all $0 < p \le \infty$ (see [P–P] or [D–L]):

$$E_{r-1}(f, J)_p \le C_r \omega_r(f, |J|, J)_p \le C_r' w_r(f, |J|, J)_p, \qquad t > 0. \tag{2.9}$$

Corollary 2.3. *If $0 < p < 1$, then for each nondecreasing $f \in L_p(I)$, $I := [-1, 1]$, and $n \ge 2$, there is a continuous piecewise linear nondecreasing spline S with n equally spaced knots (the usual notation $S \in \mathscr{S}_n^2$) such that*

$$\|f - S\|_p \le C\omega_2(f, 1/n)_p \tag{2.10}$$

with a constant C depending only on p.

Proof. We choose $\xi_j = (2j - n)/n$, $j = 0, \dots, n$, and apply Theorem 2.2 and (2.9) to the continuous piecewise linear nondecreasing S^* and find

$$\|f - S^*\|_p(I_j) \le CE_1(f, I_j^*)_p \le Cw_2(f, 14/n, I_j^*)_p. \tag{2.11}$$

We raise each of the inequalities (2.11) to the power p and then add them. Since a point $x \in I$ appears in at most seven of the I_j^* we obtain

$$\|f - S^*\|_p^p \le C \sum_{j=1}^{n} w_2(f, 14/n, I_j^*)_p^p \le Cw_2(f, 14/n, I)_p^p \le C\omega_2(f, 1/n, I)_p^p. \qquad \blacksquare$$

3. Preparatory Results

The remainder of this paper is concerned with the approximation of a given function $f \in L_p(I)$, $I := [-1, 1]$, by algebraic polynomials. For this, we shall use a very common technique. We first approximate f by a piecewise polynomial S and then approximate S by a polynomial with the desired degree and monotonicity. For example, in the simplest case, S is a piecewise linear function

$$S(x) = \sum_{j=0}^{n} c_j(x - \xi_j)_+$$

of the type considered in the previous section. The improved estimates for algebraic polynomial approximation hinge on taking the breakpoints ξ_j of the piecewise linear function S thicker near the endpoints ± 1. To construct monotone approximants, we shall need other properties for the ξ_j. We begin by recalling a construction introduced in [D–Y] and used also in [L–Y].

We shall discuss how to choose the points ξ_j and how to approximate the truncated functions $\varphi_j(x) := (x - \xi_j)_+$ by algebraic polynomials. We shall first approximate a corresponding periodic function by trigonometric polynomials, and then use the standard change of variables $x = \cos t$ to obtain algebraic polynomial approximants.

Let J_n be a Jackson kernel

$$J_n(t) = \lambda_n \left(\frac{\sin nt/2}{\sin t/2}\right)^{2r}, \qquad \int_{-\pi}^{\pi} J_n(t)\, dt = 1.$$

Here r is a fixed natural number which is chosen large enough that certain conditions (to be prescribed later) are satisfied. We begin by approximating the characteristic functions $\chi_j(t) := \chi_{[-t_j, t_j]}$, $t_j := j\pi/n$, $j = 0, \ldots, n$, by

$$T_j(t) := \chi_j * J_n(t) = \int_{t-t_j}^{t+t_j} J_n(u)\, du, \qquad j = 0, \ldots, n.$$

Making the change of variable $x = \cos t$ we obtain algebraic polynomials $r_j(x) := T_{n-j}(t)$ which are approximations to the characteristic functions $\chi_{[x_j, 1]}$, $x_j := \cos t_{n-j}$. Finally, we define

$$R_j(x) := \int_{-1}^{x} r_j(u)\, du, \qquad j = 0, \ldots, n,$$

which can be viewed as approximations to the truncated power functions φ_j. Note that $R_0(x) = 1 + x$ and $R_n(x) \equiv 0$ and, in general, R_j is a polynomial of degree $\le nr$.

For the construction of monotone algebraic polynomial approximants, we need other properties of the R_j. Note that $r_j - r_{j+1} \ge 0$, $x \in [-1, 1]$, and therefore $R_j - R_{j+1}$ is increasing in $[-1, 1]$ for all $j = 0, 1, \ldots, n-1$. We need to introduce some new points ξ_j which serve as a substitute for the x_j. They are defined by $1 - \xi_j = R_j(1)$. It follows that $-1 = \xi_0 < \xi_1 < \cdots < \xi_n = 1$. We need the following further discription of the distribution of the ξ_j's in $[-1, 1]$ which was proved in [D–Y] and says that the points ξ_j are distributed like the points $\cos t_{n-j}$.

Lemma 3.1. *With $\varphi(x) := \sqrt{1 - x^2}$ we have for any $n \ge 10$:*

(i) $C_1 \varphi(x) n^{-1} \le \xi_{j+4} - \xi_{j-3} \le C_2 \varphi(x) n^{-1}$, $x \in [\xi_{j-3}, \xi_{j+4}]$, $j = 4, \ldots, n-5$;
(ii) $|\xi_j - \cos t_{n-j}| \le C(\xi_{j+1} - \xi_j)$, $j = 0, \ldots, n-1$;
(iii) $C_1(\xi_{j+1} - \xi_j) \le \xi_j - \xi_{j-1} \le C_2(\xi_{j+1} - \xi_j)$, $j = 1, 2, \ldots, n-1$;
(iv) $1 + x \le C_2 \varphi(x) n^{-1}$, $-1 \le x \le \xi_7$, *and* $1 - x \le C_2 \varphi(x) n^{-1}$, $\xi_{n-7} \le x \le 1$;
(v) $C_1 \varphi(x) n^{-1} \le \xi_1 + 1$, $-1 \le x \le \xi_7$;

$$C_1 \varphi(x) n^{-1} \le 1 - \xi_{n-1}, \; \xi_{n-7} \le x \le 1;$$

where C, C_1, and C_2 are constants independent of n and x.

We need estimates on how close r_j and R_j are to $\theta_j := \chi_{[\xi_j, 1]}$ and $\varphi_j := (\cdot - \xi_j)_+$, respectively. This is given by

Lemma 3.2. *For $j = 1, \ldots, n-1$ let $d_j(x) := 1 + |x - \xi_j|/(\xi_{j+1} - \xi_j)$. Then*

$$|r_j(x) - \theta_j(x)| \le C[d_j(x)]^{-r+1}, \tag{3.1}$$

$$|R_j(x) - \varphi_j(x)| \le C(\xi_{j+1} - \xi_j)[d_j(x)]^{-r+2}, \tag{3.2}$$

with the constant C depending only on r.

Proof. We first let $\tilde{d}_j(t) := \max\{1, n|t - t_j|\}$ and approximate $\chi_j(t)$ by T_j. It was shown in [D–Y, Lemma 3] that for $0 \le t \le \pi$ and $a = |t - t_j|$ we have

$$|\chi_j(t) - T_j(t)| = \left| \int_{-\pi}^{\pi} [\chi_j(t) - \chi_j(t-u)] J_n(u)\, du \right|$$
$$\le \int_{|u| \ge a} J_n(u)\, du \le \int_{-\pi}^{\pi} \left| \frac{u}{a} \right|^{2r-2} J_n(u)\, du$$
$$\le C(an)^{-2r+2},$$

where we used the well-known inequalities for the moments of the Jackson kernel (see [L, p. 57]):

$$\int_{-\pi}^{\pi} |u|^s J_n(u)\, du \le Cn^{-s}, \qquad s = 0, \ldots, 2r-2.$$

Since $|\chi_j(t) - T_j(t)| \le 1$, $0 \le t \le \pi$, we have proved that

$$|\chi_j(t) - T_j(t)| \le C[\tilde{d}_j(t)]^{-2r+2}.$$

To obtain inequalities (3.1) and (3.2), we make the substitution $x = \cos t$ and proceed as in [D–Y, Lemma 5] to obtain for $x = \cos t$, $0 \le t \le \pi$,

$$|r_j(x) - \theta_j(x)| \le C[\tilde{d}_{n-j}(t)]^{-2r+2}.$$

If we integrate this inequality with respect to x, we obtain (see [D–Y, Lemma 5] or [L–Y, Lemma 6] for details)

$$|R_j(x) - \varphi_j(x)| \le C \frac{\sin t_{n-j}}{n} [\tilde{d}_{n-j}(t)]^{-2r+4}.$$

It follows from Lemma 3.1 that

$$\frac{\sin t_{n-j}}{n} \le C(\xi_{j+1} - \xi_j). \tag{3.3}$$

Hence, in order to complete the proof of (3.1) and (3.2), it suffices to prove that for $x = \cos t$, $0 \le t \le \pi$,

$$d_j(x) \le C[\tilde{d}_{n-j}(t)]^2. \tag{3.4}$$

We consider two cases. If $|t - t_{n-j}| \le \pi/n$, then $\tilde{d}_{n-j}(t) \ge 1$ while $d_j(x) \le C$ and (3.4) is obvious. Consider then the second case: $i\pi/n \le |t - t_{n-j}| \le (i+1)\pi/n$ for some $1 \le i \le n-1$. Then, for some ξ with $|\xi - t_{n-j}| \le (i+1)\pi/n$, we have

$$|\cos t - \cos t_{n-j}| = |\sin \xi||t - t_{n-j}| \le Ci|\sin t_{n-j}||t - t_{n-j}| \le Ci^2(\xi_{j+1} - \xi_j),$$

where the last inequality uses (3.3). It follows that

$$|x - \xi_j| \le |x - \cos t_{n-j}| + |\cos t_{n-j} - \xi_j| \le Ci^2(\xi_{j+1} - \xi_j).$$

Therefore $d_j(x) \le Ci^2$ while $\tilde{d}_{n-j}(t) \ge \pi i$ and (3.4) follows. ■

We shall also need some well-known inequalities for algebraic polynomials. The first of these estimates is the norm of a polynomial on a large interval in terms of its norm on a smaller interval. It follows from the well-known extremal property of Chebyshev polynomials.

Lemma 3.3. *If P is a polynomial of degree $\le k$, then for $x \in [-1, 1]$ we have*

$$|P(x)| \le C\left(1 + \frac{|x - \xi_j|}{\xi_{j+1} - \xi_j}\right)^k \max_{\xi_j \le u \le \xi_{j+1}} |P(u)|.$$

The second result we need follows from the fact that any two (quasi-)norms are equivalent on the space of polynomials of a fixed degree.

Lemma 3.4. *If $0 < p < \infty$ and $k = 0, 1, \ldots,$ then for any polynomial P of degree $\le k$ we have*

$$\max_{a \le x \le b} |P(x)|^p \le \frac{C}{b - a} \|P\|^p_{L_p[a,b]},$$

where C depends only on k and p.

4. Proof of Theorem 1.1—The Nonconstrained Case

We fix $n = 1, 2, \ldots$ and let ξ_j be the points of Section 3. Further, we denote by $I_j := [\xi_{j-1}, \xi_j]$ and as in Theorem 2.2, $I_j^* := [\xi_{j-4}, \xi_{j+3}] \cap I$, $j = 1, \ldots, n-1$, where $\xi_j := \xi_0 = -1, j \le 0$, and $\xi_j := \xi_n = 1, j \ge n$. For $f \in L_p(I)$ we denote by $p_j(f)$ an algebraic polynomial of degree $\le k - 1$ which is a near-best approximation to f on the interval I_j^*. We should remark at the outset that if we wish only to discuss unconstrained approximation, we could work with the intervals I_j in place of I_j^*. However, we wish to give a proof which applies in the case of monotone approximation as well.

For $j = 1, \ldots, n - 1$ we obtain from Whitney's theorem (2.9) and the equivalence of ω_k and w_k (see (2.8)),

$$\begin{aligned}(4.1)\quad \|f - p_j\|^p_p(I_j) &\le C\omega_k(f, |I_j|, I_j)^p_p \\ &\le Cw_k(f, |I_j^*|, I_j^*)^p_p \le C|I_j^*|^{-1} \int_0^{|I_j^*|} \int_{I_j^*} |\Delta_h^k(f, x, I_j^*)|^p \, dx \, dh \\ &= C \int_{I_j^*} \int_0^{|I_j^*|/\varphi(x)} \frac{\varphi(x)}{|I_j^*|} |\Delta^k_{h\varphi(x)}(f, x, I_j^*)|^p \, dh \, dx,\end{aligned}$$

where we changed the order of integration and made a simple change of variables to arrive at the last integral.

Let

$$(4.2)\qquad L_n(f, x) = p_1(f, x) + \sum_{j=1}^{n-1} [p_{j+1}(f, x) - p_j(f, x)]\theta_j(x),$$

and

$$P_n(f, x) = p_1(f, x) + \sum_{j=1}^{n-1} [p_{j+1}(f, x) - p_j(f, x)]r_j(x), \tag{4.3}$$

where θ_j and r_j are defined as in Section 3.

We shall see that the polynomial P_n has the desired approximation properties. We first prove

Theorem 4.1. *If $f \in L_p(I)$, $0 < p < 1$, and k is a positive integer, then*

$$\|f - L_n(f)\|_p \le C\omega_k^\varphi\left(f, \frac{1}{n}\right)_p, \qquad n \ge 10, \tag{4.4}$$

where C depends only on k and p.

Proof. We fix $n \ge 10$. For $j = 5, \dots, n-4$ we have by Lemma 3.1(i) that $C_1 n^{-1} \le |I_j^*|/\varphi(x) \le C_2 n^{-1}$ for $x \in I_j^*$. Since $L_n(f) = p_j(f)$ on I_j, from (4.1) we obtain

$$\begin{aligned} \int_{I_j} |f(x) - L_n(f, x)|^p\,dx &= \int_{I_j} |f(x) - p_j(f, x)|^p\,dx \\ &\le Cn \int_0^{C_2/n} \int_{I_j^*} |\Delta_{h\varphi(x)}^k(f, x, I_j^*)|^p\,dx\,dh. \end{aligned} \tag{4.5}$$

This same inequality also holds for $j = 1, 2, 3, 4$ and $j = n-3, n-2, n-1, n$. For example, for $j = 1, 2, 3, 4$ we have $\Delta_{h\varphi(x)}^k(f, x, I_j^*) = 0$ if $x \le -1 + kh\varphi(x)/2$, that is, if $h \ge 2(x+1)/k\varphi(x)$. This means that the inner integral on the right side of (4.1) can be taken over $0 \le h \le 2(x+1)/k\varphi(x) \le Cn^{-1}$ by Lemma 3.1(iv). Also, by Lemma 3.1(v), $\varphi(x)/|I_j^*| \le Cn$. Hence, we again obtain (4.5). The other values of j are handled in the same way.

We now sum the inequalities (4.5) to find

$$\int_I |f(x) - L_n(f, x)|^p\,dx \le Cn \int_0^{C_2/n} \int_I |\Delta_{h\varphi(x)}^k(f, x, I)^p|\,dx\,dh \le C\omega_k^\varphi\left(f, \frac{1}{n}\right)_p^p, \tag{4.6}$$

where we have used the fact that a point $x \in I$ appears in at most seven of the intervals I_j^*, $j = 1, \dots, n$. ■

We are ready to prove Theorem 1.1 in the nonconstrained case.

Proof of Theorem 1.1 (Part 1). In view of Theorem 4.1 and the fact that $P_n(f, x)$, which is defined in (4.3), is a polynomial of degree $\le (n-1)r + k$, we only need to prove that

$$\int_{-1}^{1} |L_n(f, x) - P_n(f, x)|^p\,dx \le C\omega_k^\varphi\left(f, \frac{1}{n}\right)_p^p. \tag{4.7}$$

By Lemmas 3.2 and 3.3 we have

$$(4.8)\quad I' := \int_{-1}^{1} |L_n(f, x) - P_n(f, x)|^p \, dx$$
$$= \int_{-1}^{1} \left| \sum_{i=1}^{n-1} (p_{i+1}(f, x) - p_i(f, x))(\theta_i(x) - r_i(x)) \right|^p dx$$
$$\leq \sum_{i=1}^{n-1} \int_{-1}^{1} |p_{i+1}(f, x) - p_i(f, x)|^p \, |\theta_i(x) - r_i(x)|^p \, dx$$
$$\leq C \sum_{i=1}^{n-1} \max_{x \in I_{i+1}} |p_{i+1}(f, x) - p_i(f, x)|^p \int_{-1}^{1} \left(1 + \frac{|x - \xi_i|}{\xi_{i+1} - \xi_i}\right)^{(k-1)p}$$
$$\times |\theta_i(x) - r_i(x)|^p \, dx$$
$$\leq C \sum_{i=1}^{n-1} \max_{x \in I_{i+1}} |p_{i+1}(f, x) - p_i(f, x)|^p \int_{-1}^{1} \left(1 + \frac{|x - \xi_i|}{\xi_{i+1} - \xi_i}\right)^{kp - rp} dx,$$

where $I_i := [\xi_{i-1}, \xi_i]$.

Now we choose r so that $rp - kp > 2$ and it is readily seen that

$$(4.9)\qquad \int_{-1}^{1} [d_i(x)]^{-2} \, dx \leq C(\xi_{i+1} - \xi_i), \qquad i = 1, \ldots, n-1.$$

Hence by Lemma 3.4, (4.5), and (4.8) we obtain

$$(4.10)\qquad I' \leq C \sum_{i=1}^{n-1} \max_{x \in I_{i+1}} |p_{i+1}(f, x) - p_i(f, x)|^p |\xi_{i+1} - \xi_i|$$
$$\leq C \sum_{i=1}^{n-1} \int_{\xi_i}^{\xi_{i+1}} |p_{i+1}(f, x) - p_i(f, x)|^p \, dx$$
$$\leq C \sum_{i=1}^{n} \int_{\xi_{i-1}}^{\xi_{i+1}} |f(x) - p_i(f, x)|^p \, dx$$
$$\leq C \omega_k^{\varphi}\left(f, \frac{1}{n}\right)_p^p.$$

Here, the last inequality uses that each integral in the last sum can be estimated as in (4.5) because of (4.1). This completes the proof of Theorem 1.1 in the nonconstrained case. ■

5. Proof of Theorem 1.1—The Monotone Case

Here we use the continuous piecewise linear function S^* of Section 2. For $k = 2$, we can take $p_j(f)$ in Section 4 to be the $l_j^* := a_j x + b_j$ of Theorem 2.2. Let

$$(5.1)\qquad L_n^*(f, x) = p_1(f, -1) + \sum_{j=0}^{n-1} a_{j+1}(\varphi_j(x) - \varphi_{j+1}(x))$$

and

$$P_n^*(f, x) = p_1(f, -1) + \sum_{j=0}^{n-1} a_{j+1}(R_j(x) - R_{j+1}(x)). \tag{5.2}$$

We note that since $R_j - R_{j+1}$ is increasing for $j = 0, \ldots, n-1$ and $a_j \geq 0$ for $j = 1, \ldots, n$, the polynomial $P_n^*(f, x)$ is nondecreasing in $[-1, 1]$. Also, because $p_{i+1}(f, \xi_i) = p_i(f, \xi_i)$, $i = 1, \ldots, n-1$, it follows that

$$\begin{aligned} L_n^*(f, x) &= p_1(f, -1) + a_1(1 + x) + \sum_{j=1}^{n-1} (a_{j+1} - a_j)\varphi_j(x) \\ &= p_1(f, x) + \sum_{j=1}^{n-1} (a_{j+1} - a_j)(x - \xi_j)\theta_j(x) \\ &= p_1(f, x) + \sum_{j=1}^{n-1} [p_{j+1}(f, x) - p_j(f, x)]\theta_j(x) \\ &= L_n(f, x). \end{aligned}$$

We need to estimate

$$\begin{aligned} &\int_{-1}^{1} |f(x) - P_n^*(f, x)|^p \, dx \\ &\quad \leq \int_{-1}^{1} |f(x) - L_n(f, x)|^p \, dx + \int_{-1}^{1} |L_n^*(f, x) - P_n^*(f, x)|^p \, dx. \end{aligned}$$

So in view of Theorem 4.1, we have only to estimate the second term. To this end,

$$\begin{aligned} |L_n^*(f, x) - P_n^*(f, x)| &\leq \sum_{i=1}^{n-1} |a_{i+1} - a_i| |\varphi_i(x) - R_i(x)| \\ &= \sum_{i=1}^{n-1} \frac{|p_{i+1}(f, \xi_{i+1}) - p_i(f, \xi_{i+1})|}{\xi_{i+1} - \xi_i} |\varphi_i(x) - R_i(x)|. \end{aligned}$$

Hence, by Lemmas 3.2 and 3.3 we have, as in the derivation of (4.10),

$$\begin{aligned} &\int_{-1}^{1} |L_n^*(f, x) - P_n^*(f, x)|^p \, dx \\ &\quad \leq \sum_{i=1}^{n-1} |p_{i+1}(f, \xi_{i+1}) - p_i(f, \xi_{i+1})|^p (\xi_{i+1} - \xi_i)^{-p} \int_{-1}^{1} |\varphi_i(x) - R_i(x)|^p \, dx \\ &\quad \leq C \sum_{i=1}^{n-1} \max_{x \in I_{i+1}} |p_{i+1}(f, x) - p_i(f, x)|^p \int_{-1}^{1} [d_i(x)]^{-(r-2)p} \, dx \\ &\quad \leq C \sum_{i=1}^{n-1} \max_{x \in I_{i+1}} |p_{i+1}(f, x) - p_i(f, x)|^p |\xi_{i+1} - \xi_i| \\ &\quad \leq C\omega_2^{\varphi}\left(f, \frac{1}{n}\right)_p^p, \end{aligned}$$

provided $rp - 2p > 2$.

This completes the proof of Theorem 1. ∎

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