CONVEX POLYNOMIAL AND SPLINE APPROXIMATION IN L_p, $0 < p < \infty$

R. A. DeVore, Y. K. Hu and D. Leviatan[1]

Abstract. We prove that a convex function $f \in L_p[-1,1]$, $0 < p < \infty$, can be approximated by convex polynomials with an error not exceeding $C\omega_3^\varphi(f, \frac{1}{n})_p$ where $\omega_3^\varphi(f,\cdot)$ is the Ditzian-Totik modulus of smoothness of order three of f. We are thus filling the gap between previously known estimates involving $\omega_2^\varphi(f,1/n)_p$, and the impossibility of having such estimates involving ω_4. We also give similar estimates for the approximation of f by convex C^0 and C^1 piecewise quadratics as well as convex C^2 piecewise cubic polynomials.

1. Introduction and main results. Let $f \in L_p[-1,1]$, $0 < p \le \infty$ (where by L_∞ we mean $C[-1,1]$) be a convex function. We are interested in estimating the degree of approximation of f in the L_p-(quasi-)norm by means of convex polynomials or convex splines.

The first estimates of this type are due to Švedov [SVE] who proved that for a given convex $f \in L_p[-1,1]$, $0 < p \le \infty$, and $n \ge 2$, there exist convex polynomials p_n of degree not exceeding n, such that

$$\|f - p_n\|_p \le C\omega_2(f, 1/n)_p, \tag{1.1}$$

where $C = C(p)$ is an absolute constant independent of f and n, and $\omega_2(f,\cdot)_p$ is the ordinary second order modulus of smoothness in the L_p norm. Švedov [SVE] went on to prove that in (1.1), ω_2 cannot be replaced by ω_4 while keeping the constant independent of f and n.

In recent years (1.1) has been improved in a sequence of papers by DeVore, Leviatan and Yu (see [DLE], [LY] and [Y]) who were able to replace ω_2 by the Ditzian-Totik second modulus of smoothness in L_p. Namely, they proved that for a convex $f \in L_p[-1,1]$, $0 < p < \infty$, and each $n \ge 1$, there exist convex polynomials p_n of degree $\le n$ such that

$$\|f - p_n\|_p \le C\omega_2^\varphi(f, 1/n)_p. \tag{1.2}$$

While (1.2) is also valid for $p = \infty$, better estimates are now known in that case. In fact for a convex $f \in \mathbf{C}[-1,1]$ there exist convex polynomials p_n such that

$$\|f - p_n\|_{\mathbf{C}[-1,1]} \le C\omega_3^\varphi(f, 1/n)_\infty. \tag{1.3}$$

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(See [KOP] and see [HLY] for a weaker but earlier result.) One of the main aims of the present paper is to show that a similar estimate to (1.3) involving $\omega_3^\varphi(f,\cdot)_p$, $0<p<\infty$, is also valid, thus completely closing the gap left by Švedov [SVE]. To this end we prove the following result in §3.

Theorem 1.1. *Let $f \in L_p[-1,1]$, $0<p<\infty$, be convex. Then for each $n \geq 2$, there is a convex polynomial p_n such that*

$$\|f - p_n\|_p \leq C\omega_3^\varphi(f, \frac{1}{n})_p, \tag{1.4}$$

where C is a constant which depends at most on p when $p \to 0$.

Our proof of Theorem 1.1, given in §3, is based in part on a convex, C^0 piecewise polynomial approximation to f which is constructed in §2.

In §4, we consider piecewise polynomial approximation a little further. To describe these results, we let $E(I) := E(f,I)_p$ denote the error in L_p approximation of f on the interval I by quadratic polynomials. We recall that from Whitney's theorem [DLO, p.182, p.374] , we have

$$E(I) \leq C\omega_3(f,|I|,I)_p. \tag{1.5}$$

for all intervals I with a constant C depending only on p as $p \to 0$.

Now, let $\mathbf{T}_n := \{-1 =: t_0 < t_1 < \cdots < t_n := 1\}$, $n \geq 1$, be any partition of $[-1,1]$, and set $t_j := -1$, $j<0$, and $t_j := 1$, $j>n$. For $j=-1,\ldots,n$, let $I_j := [t_j, t_{j+1}]$, $I_j^{(1)} := [t_{j-11}, t_{j+11}]$, and $I_i^{(2)} := [t_{j-49}, t_{j+49}]$. Then we have,

Theorem 1.2. *Let $f \in L_p[-1,1]$, $0<p\leq\infty$, be convex. Then there exists a convex C^1 piecewise quadratic polynomial $S_{(1)}$ and a convex C^2 piecewise cubic polynomial $S_{(2)}$ on the partition $\mathbf{T}_n$, such that for $k=1,2$,*

$$\left\|f - S_{(k)}\right\|_{L_p(I_j)} \leq CE(I_j^{(k)})_p, \quad j=0,\ldots,n-1, \tag{1.6}$$

where $|J|$ denotes the length of the interval J and C depends on $\max_{j=0}^{n-1}|I_{j\pm1}|/|I_j|$, and also on p as $p \to 0$.

The proof of Theorem 1.2 is given §4. It utilizes the C^0 piecewise quadratic approximation constructed in §2 together with some smoothing techniques. The latter uses ideas of Ivanov and Popov [IP] for smoothing piecewise polynomial approximants while preserving local approximation errors. Ivanov and Popov used a similar technique to smooth a continuous piecewise quadratic with equidistant knots so that it preserves convexity and stays close to the original function in the sup-norm. The L_p estimates are based on ideas from [HLY].

The most interesting partitions for piecewise polynomial approximation are the uniform partition and the partitions built on the zeros of the Tchebyshev polynomials. In both cases the ratios $|I_{j\pm1}|/|I_j|$ are bounded independent of n. Hence we have absolute constants independent of n, in the above theorems.

2. Convex piecewise quadratic approximation. In this section, we shall consider the $L_p[-1,1]$, $0 < p \le \infty$, approximation of a convex function f by piecewise quadratics. We fix p and fix the function f throughout this section. For any interval I, let $E(I) := E(f, I)_p$ be the error in approximating f in the metric $L_p(I)$ by quadratic polynomials. We say that a quadratic polynomial q is a near best L_p-approximation to f on I with constant C_0 if

$$\|f - q\|_{L_p(I)} \le C_0 E(I).$$

We shall often make use of the following facts (for proofs of similar results see [DSH, Ch. 3]).

F1. If q is a near best approximation to f on I with constant C_0, then for any interval J with $I \subset J$, q is also a near best approximation to f on J with constant C depending only on $|J|/|I|$, C_0, and p as $p \to 0$.

F2. Let $I \subset J$ be two intervals. If q is a quadratic polynomial satisfying

$$\|f - q\|_{L_p(I)} \le C_0 E(J),$$

then

$$\|f - q\|_{L_p(J)} \le C E(J)$$

with the constant C depending on on $|J|/|I|$, C_0, and p as $p \to 0$.

F3. For any quadratic polynomial q and any interval J,

$$\|q\|_{L_p(J)} \le |J|^{1/p} \|q\|_{L_\infty(J)} \le C \|q\|_{L_p(J)}$$

with C depending only on p as $p \to 0$.

Let $\mathbf{Z}_n := \{-1 =: z_0 < z_1 < \cdots < z_n := 1\}$ be a given partition of $[-1,1]$ and extend this partition by setting $z_j := -1$, $j < 0$, and $z_j := 1$, $j > n$. We let $J_j := [z_j, z_{j+1}]$, $j = 0, \dots, n-1$. In this section, we shall construct various piecewise quadratic approximations to f which share the convexity of f. We begin by singling out some special points ξ_j near z_j, $j = 0, \dots, n$.

Let $\delta_j := \frac{1}{3}|J_j|$, $j = 0, \dots, n-1$, and $\delta_n := \frac{1}{3}|J_{n-1}|$, and let $\hat{J}_j := [z_j, z_j + \delta_j]$, $j = 0, \dots, n-1$, $\hat{J}_n := [z_n - \delta_n, z_n]$. Furthermore, let $\bar{J}_j := [z_{j-2}, z_{j+3}]$, $j = 0, 1, \dots, n$.

Throughout this section, we shall use C to denote a constant which depends only on $\max_{j=0}^{n-1} |J_{j\pm1}|/|J_j|$, and p as $p \to 0$. We begin with the following simple lemma.

Lemma 2.1. *There are points $\xi_j \in \hat{J}_j$, $j = 0, \dots, n$, such that the polynomial q_j which interpolates f at ξ_i, $i = j-1, j, j+1$, is a near best approximation to f on $\bar{J}_j$:*

$$\|f - q_j\|_{L_p(\bar{J}_j)} \le C E(\bar{J}_j), \quad j = 1, \dots, n-1.$$

Proof. We begin with the best quadratic polynomial approximation p_j to f on J_j, $j = 0, \dots, n-1$ and we put $p_{-1} := p_0$, and $p_{n+1} := p_n := p_{n-1}$. Then,

$$\|f - p_j\|_{L_p(J_j)} = E(J_j), \quad j = 0, \dots, n-1. \tag{2.1}$$

It follows that

$$\|f - p_{j-1}\|_{L_p(\hat{J}_j)} + \|f - p_j\|_{L_p(\hat{J}_j)} + \|f - p_{j+1}\|_{L_p(\hat{J}_j)} \le CE([z_{j-1}, z_{j+2}]), \quad j = 0, \ldots, n.$$

Hence, we can find a point $\xi_j \in \hat{J}_j$, $j = 0, \ldots, n$, such that for each $j = 0, \ldots, n$,

$$|f(\xi_j) - p_i(\xi_j)| \le C|\hat{J}_j|^{-1/p} E([z_{j-1}, z_{j+2}]), \quad i = j-1, j,, j+1. \tag{2.2}$$

Now let q_j be the quadratic polynomial which interpolates f at the points ξ_i, $i = j-1, j, j+1$. Then, from (2.2), for $j = 1, \ldots, n-1$,

$$|q_j(\xi_i) - p_j(\xi_i)| \le C|\bar{J}_j|^{-1/p} E(\bar{J}_j), \quad i = j-1, j, j+1.$$

Considering the spacing of the points ξ_i, we see that

$$\|q_j - p_j\|_{L_p(\bar{J}_j)} \le C|\bar{J}_j|^{1/p} \|q_j - p_j\|_{L_\infty(\bar{J}_j)} \le C|\bar{J}_j|^{1/p} \max_{i=j-1,j,j+1} |q_j(\xi_i) - p_j(\xi_i)| \le CE(\bar{J}_j).$$

Hence,

$$\|f - q_j\|_{L_p(\bar{J}_j)} \le C\left(\|f - p_j\|_{L_p(\bar{J}_j)} + \|p_j - q_j\|_{L_p(\bar{J}_j)}\right) \le CE(\bar{J}_j). \quad \square$$

We will build a continuous piecewise quadratic from the polynomials q_j. For this, we shall use the following lemma.

Lemma 2.2. *Let $-1 \le \zeta_0 < \zeta_1 < \cdots < \zeta_n \le 1$ be arbitrary points and let f be a convex function on $[-1, 1]$. For each $j = 1, \ldots, n-1$, let $\tilde{p}_j$ be the quadratic polynomial which interpolates f at the points ζ_i, $i = j-1, j, j+1$. Then the function*

$$\tilde{g}(x) := \begin{cases} \tilde{p}_1(x), & x \in [-1, \zeta_1] \\ \max(\tilde{p}_j(x), \tilde{p}_{j+1}(x)), & x \in [\zeta_j, \zeta_{j+1}], \quad j = 1, \ldots, n-2 \\ \tilde{p}_{n-1}(x), & x \in [\zeta_{n-1}, 1]. \end{cases} \tag{2.3}$$

is a convex C^0 piecewise quadratic with breakpoints at the ζ_j, $j = 1, \ldots, n-1$.

Proof. See K. A. Kopotun [KOP] and K. G. Ivanov and B. Popov [IP] where similar constructions have been used. $\square$

We apply Lemma 2.2 to the polynomials q_j and the points $\{\xi_j\}$ and denote by g_0 the resulting C^0 piecewise quadratic. We note that on each interval $[\xi_j, \xi_{j+1}]$, the function g_0 is identical with one of the polynomials q_j or q_{j+1}.

Now g_0 has many of the desired properties but its breakpoints ξ_j are (near but) not exactly the z_j. We are going to modify g_0 to obtain a piecewise quadratic g which has all the desired approximation properties and, in addition, has its break points exactly at the z_j. We let r_j be the (convex) quadratic which interpolates g_0 at z_i, $i = j-1, j, j+1$, $j = 1, \ldots, n-1$. We apply Lemma 2.2 to the polynomials r_j and the interpolation points z_j to obtain a C^0 piecewise quadratic g with breakpoints at the z_j, $j = 1, \ldots, n-1$.

Theorem 2.3. *Let $f \in L_p[-1,1]$, $0 < p \le \infty$, be convex. Then, the continuous convex piecewise quadratic g on the partition $\mathbf{Z}_n$ defined above satisfies*

$$\|f - g\|_{L_p(J_j)} \le CE([z_{j-4}, z_{j+5}]), \quad j = 0, \dots, n-1. \tag{2.4}$$

Proof. On the interval J_j, g is identical with r_j or with r_{j+1}. In the case $j = 0$, this choice is r_1 and in the case $j = n-1$, this choice is r_{n-1}. We shall first estimate $\|f - r_j\|_{L_p(J_j)}$, $1 \le j \le n-1$. From Lemma 2.1 we have

$$\begin{aligned} \|f - r_j\|_{L_p(J_j)} &\le C\left(\|f - q_j\|_{L_p(J_j)} + \|q_j - r_j\|_{L_p(J_j)}\right) \\ &\le C\left(E(\bar{J}_j) + \|q_j - r_j\|_{L_p(J_j)}\right). \end{aligned} \tag{2.5}$$

For the the sake of convenience in notation in the proof below, we need to set $q_n := q_{n-1}$ and $q_{-1} := q_0 := q_1$. Now recall that at each point z_i, $i = j-1, j, j+1$, r_j interpolates the value $g_0(z_i)$ and this value is the same as either $q_{i-1}(z_i)$ or $q_i(z_i)$. Therefore, from Lemma 2.1,

$$\begin{aligned} \|q_j - r_j\|_{L_p(J_j)} &\le C|J_j|^{-1/p}\,\|q_j - r_j\|_{L_\infty(J_j)} \le C|J_j|^{-1/p} \max_{i=j-1,j,j+1} |q_j(z_i) - r_j(z_i)| \\ &\le C|J_j|^{-1/p} \max_{j-2 \le k \le j+1} \|q_j - q_k\|_{L_\infty[z_{j-1}, z_{j+1}]} \\ &\le C \max_{j-2 \le k \le j+1} \|q_j - q_k\|_{L_p[z_{j-1}, z_{j+1}]} \\ &\le C(\|f - q_j\|_{L_p[z_{j-1}, z_{j+1}]} + \max_{j-2 \le k \le j+1} \|f - q_k\|_{L_p[z_{j-1}, z_{j+1}]} \\ &\le C(E(\bar{J}_j) + \max_{j-2 \le k \le j+1} E(\bar{J}_k)) \le E([z_{j-4}, z_{j+4}]). \end{aligned} \tag{2.6}$$

Using this in (2.5) shows that for $j = 1, \dots, n-1$,

$$\|f - r_j\|_{L_p(J_j)} \le CE([z_{j-4}, z_{j+4}]). \tag{2.7}$$

The same analysis with r_j replaced by r_{j+1} gives for $j = 0, \dots, n-2$,

$$\|f - r_{j+1}\|_{L_p(J_j)} \le CE([z_{j-3}, z_{j+5}]).$$

We have therefore proved (2.4).

$\square$

3. Construction of convex polynomials. In this section, we shall prove Theorem 1.1 on approximation by convex polynomials. We fix the convex function f and $0 < p < \infty$. To prove Theorem 1.1 we shall first approximate f by a convex C^0 piecewise quadratic g given by Theorem 2.3 and then approximate g by an algebraic polynomial. This method of proof has been used several times before but we shall follow most closely the constructions and notation in Kopotun [KOP] (although our notation does vary slightly from Kopotun especially in the ordering of the knots).

Let $x_{n,j} := \cos \frac{(n-j)}{n}\pi$, $j = 0, \ldots, n$, $x_{n,j} := -1$, $j < 0$, and $x_{n,j} := 1$, $j > n$. We shall apply the results of §2 with the $z_j = x_{n,j}$, for all j. We let let $J_{n,j}$, $\bar{J}_{n,j}$ etc., be the obvious analogues of the intervals in that section. Note that the ratios $|J_{n,j\pm1}|/|J_j|$ are bounded by a constant independent of n and j, so that the estimates of §2 hold with absolute constants (except for the dependence on p).

We shall use some results of Kopotun [KOP] on the approximation of truncated powers. To describe these, we let $\chi_{n,j} := \chi_{[x_{n,j},1]}$ and $\psi_{n,j} := \frac{|J_{n,j}|}{|x-x_{n,j}|+|J_{n,j}|}$, $1 \le j \le n-1$. Then Kopotun [KOP, Lemma 2] has constructed three sets of polynomials $\sigma_{n,j}$, $R_{n,j}$ and $\overline{R}_{n,j}$, $j = 1, \ldots, n-1$, of degrees not exceeding some fixed multiple of n, which approximate the truncated powers. Kopotun's applications were primarily in the L_∞-norm. However, we shall need estimates in the L_p-norm, and as $p \to 0$, we need better and better estimates. The following estimates can be obtained from Kopotun's proof on closer look. We shall not indicate the dependence of the polynomials on p and all constants C depend at most on p as $p \to 0$. The polynomials $\sigma_{n,j}$, $R_{n,j}$ and $\overline{R}_{n,j}$, $j = 1, \ldots, n-1$ are of degree not exceeding $\max\{50, \frac{1}{p}\}n$ and satisfy the following estimate:

$$|(x - x_{n,j})_+ - \sigma_{n,j}(x)| \le C\psi_{n,j}^{\max\{1,1/p\}16}|J_{n,j}|, \qquad x \in [-1,1] \tag{3.1}$$

$$|(x - x_{n,j})_+^2 - R_{n,j}(x)| \le C\psi_{n,j}^{\max\{1,1/p\}16}|J_{n,j}|^2, \qquad x \in [-1,1] \tag{3.2}$$

and

$$|(x - x_{n,j})_+^2 - \overline{R}_{n,j}(x)| \le C\psi_{n,j}^{\max\{1,1/p\}16}|J_{n,j}|^2, \qquad x \in [-1,1]. \tag{3.3}$$

Also, it is possible to prescribe an (sufficiently large) integer M so that for all n and all $j = 1, \ldots, n-1$,

$$(x_{n,j+1} - x_{n,j})\sigma''_{Mn,Mj}(x) - R''_{Mn,Mj}(x) \ge -2\chi_{n,j}(x), \qquad x \in [-1,1], \tag{3.4}$$

and

$$(x_{n,j} - x_{n,j-1})\sigma''_{Mn,Mj}(x) + \overline{R}''_{Mn,Mj}(x) \ge 2\chi_{n,j}(x), \qquad x \in [-1,1]. \tag{3.5}$$

Note that

$$|J_{n,j}| \sim |J_{Mn,Mj}|, \tag{3.6}$$

with constants in this equivalence that are independent of n and $i = 0, \ldots, n-1$.

Now let g_n be the function of Theorem 2.3 for f and the points $\{x_{n,j}\}_{j=0}^n$. We can represent g_n as a sum of the truncated powers $(x - x_{n,j})_+$ and $(x - x_{n,j})_+^2$, $j = 1, \ldots, n$. As in Kopotun [KOP], we shall classify the knots $x_{n,j}$ according to four types depending on the second order divided differences of g_n:

$$a_{n,j} := [x_{n,j-1}, x_{n,j}, x_{n,j+1}]g_n, \quad j = 1, \ldots, n-1. \tag{3.7}$$

For the following comparisons, we define $a_{n,0} = a_{n,n} := \infty$

Let $1 \le j \le n-1$. We define $x_{n,j}$ to be of type I, if

$$a_{n,j+1} < a_{n,j} \le a_{n,j-1}. \tag{3.8}$$

We define $x_{n,j}$ to be of type II, if

$$a_{n,j-1} < a_{n,j} \le a_{n,j+1}, \tag{3.9}$$

We define $x_{n,j}$ to be of type III, if

$$\max\{a_{n,j-1}, a_{n,j+1}\} < a_{n,j}, \tag{3.10}$$

All other $x_{n,j}$'s are defined to be of type IV. Note that $x_{n,1}$ can only be of type I or type IV, and $x_{n,n-1}$ can only be of type II or type IV.

In order to represent g_n as a sum of truncated powers we define

$$A_{n,j} := a_{n,j} - a_{n,j+1}, \quad j = 1, \ldots, n-2, \quad B_{n,j} := -A_{n,j-1}, \quad j = 2, \ldots, n-1,$$

and

$$A_{n,0} := [x_{n,0}, x_{n,1}]g - [x_{n,1}, x_{n,2}]g + [x_{n,0}, x_{n,2}]g.$$

It follows from the definitions of type that $A_{n,j} > 0$ for $x_{n,j}$ of types I or III and $B_{n,j} > 0$ if $x_{n,j}$ is of types II or III.

Using the divided difference representation of polynomials, we obtain (see [KOP]) the following representation for g_n for $x \in [-1, 1]$:

$$\begin{aligned} g_n(x) &= g(-1) + A_{n,0}(x+1) + a_{n,1}(x+1)^2 \\ &\quad + \sum_{x_{n,j} \in I \cup III} A_{n,j}\left((x_{n,j+1} - x_{n,j})(x - x_{n,j})_+ - (x - x_{n,j})_+^2\right) \\ &\quad + \sum_{x_{n,j} \in II \cup III} B_{n,j}\left((x_{n,j} - x_{n,j-1})(x - x_{n,j})_+ + (x - x_{n,j})_+^2\right). \end{aligned} \tag{3.11}$$

We shall next estimate $A_{n,j}$. We fix n and let $g = g_n$ and $z_j := x_{n,j}$, for all j. Let p_j be the best $L_p[z_{j-1}, z_{j+2}]$ approximation to f by quadratic polynomials. We recall, that on each interval $J_j := [z_j, z_{j+1}]$, g is either r_j or r_{j+1} with the r_j the polynomials of §2. Hence,

$$\begin{aligned} |A_{n,j}| &= (z_{j+2} - z_{j-1})|[z_{j-1}, z_j, z_{j+1}, z_{j+2}](g - p_j)| \\ &\le C|J_j|^{-2}\|g - p_j\|_{L_\infty[z_{j-1}, z_{j+2}]} \le C|J_j|^{-2} \max_{j-1 \le i \le j+2} \|r_i - p_j\|_{L_\infty[z_{j-1}, z_{j+2}]} \\ &\le C|J_j|^{-2-1/p} \max_{j-1 \le i \le j+2} \|r_i - p_j\|_{L_p[z_{j-1}, z_{j+2}]}. \end{aligned}$$

We write $r_i - p_j = f - p_j - (f - r_i)$ and use (2.7) to obtain

$$|A_{n,j}| \le C|J_j|^{-2-1/p} E([z_{j-5}, z_{j+6}]),$$

where the constant C depends at most on p as $p \to 0$. With the notation $J_{j,n}^* := [x_{n,j-6}, x_{n,j+6}]$, we obtain

$$|A_{n,j}|, |B_{n,j}| \le C|J_j|^{-2-1/p} E(J_{j,n}^*). \tag{3.12}$$

As our approximation to f, we take the polynomial

$$\begin{aligned}P_n(x) &:= g(-1) + A_{n,0}(x+1) + a_{n,1}(x+1)^2 \\ &\quad + \sum_{x_{n,j}\in I\cup III} A_{n,j}\left((x_{n,j+1}-x_{n,j})\sigma_{Mn,Mj}(x) - R_{Mn,Mj}(x)\right) \\ &\quad + \sum_{x_{n,j}\in II\cup III} B_{n,j}\left((x_{n,j}-x_{n,j-1})\sigma_{Mn,Mj}(x) + \overline{R}_{Mn,Mj}(x)\right).\end{aligned}$$

Note that P_n is a polynomial of degree at most $Mn\max(50,1/p)$ and that we keep the $A_{n,j}$'s and $B_{n,j}$'s and the knots $x_{n,j}$ at level n while taking the polynomials σ, R and $\overline{R}$ at the level Mn. However, one should observe that $x_{n,j} = x_{Mn,Mj}$ for all $j = 0,\dots,n$. Now by virtue of (3.4) and (3.5), it follows that

$$P_n''(x) \ge g_n''(x) \ge 0, \quad \text{for all} \quad x\in[-1,1], \quad x\ne x_{n,j}, \quad 1\le i\le n-1.$$

Thus P_n is convex on $[-1,1]$.

We next estimate $\|g_n - P_n\|_{L_p[-1,1]}$ in the case $1\le p<\infty$. We claim

$$\begin{aligned}&\|g_n-P_n\|^p_{L_p[-1,1]} \\ &= \int_{-1}^{1}\Bigg|\sum_{x_{n,j}\in I\cup III} A_{n,j}\big[(x_{n,j+1}-x_{n,j})\left((x-x_{n,j})_+ - \sigma_{Mn,Mj}(x)\right) \\ &\qquad - \left((x-x_{n,j})_+^2 - R_{Mn,Mj}(x)\right)\big] \\ &+ \sum_{x_{n,j}\in II\cup III} B_{n,j}\big[(x_{n,j}-x_{n,j-1})\left((x-x_{n,j})_- \sigma_{Mn,Mj}(x)\right) \\ &\qquad + \left((x-x_{n,j})_+^2 - \overline{R}_{Mn,Mj}(x)\right)\big]\Bigg|^p dx \qquad (3.13) \\ &\le C\int_{-1}^{1}\Bigg[\sum_{x_{n,j}\in I\cup III}|A_{n,j}||J_{Mn,Mj}|^2\psi^{16}_{Mn,Mj}(x) \\ &\qquad + \sum_{x_{n,j}\in II\cup III}|B_{n,j}||J_{Mn,Mj}|^2\psi^{16}_{Mn,Mj}(x)\Bigg]^p dx \\ &\le C\sum_{j=1}^{n-1}|J_{n,j}|^{-1}E(J^*_{n,j})^p\int_{-1}^{1}\psi^{16}_{Mn,Mj}(x)\,dx \le C\sum_{j=1}^{n-1}E(J^*_{n,j})^p,\end{aligned}$$

Indeed, the first inequality in (3.13) uses (3.1), (3.2) and (3.3), the second inequality uses Jensen's inequality and our estimates (3.12) for the $A_{n,j}$ and the $B_{n,j}$, and the last inequality uses the straightforward estimate

$$\int_{-1}^{1}\psi^{16}_{n,j}\,dx \le C|J_{n,j}|.$$

The estimate (3.13) also holds for $0 < p < 1$ with the only change in the proof is that one uses the subadditivity of $\|\cdot\|^p_{L_p[-1,1]}$.

Analogous to (3.13), we obtain the estimate

$$\|f - g_n\|^p_{L_p[-1,1]} \le C \sum_{j=1}^{n-1} E(J^*_{n,j})^p, \tag{3.14}$$

by adding the local estimates (2.4).

Therefore, we have

$$\|f - P_n\|^p_{L_p[-1,1]} \le C \sum_{j=1}^{n-1} E(J^*_{n,j})^p, \tag{3.15}$$

By Whitney's theorem (1.5), we have

$$E(J) \le C\omega_3(f, |J|; J)_p \tag{3.16}$$

for each interval J and with a constant depending only on p as $p \to 0$. Also, it is proved in [DLY] (see e.g. (4.1) and (4.5) there) that for any interval $J \in [-1,1]$

$$\begin{aligned} \omega_3(f, |J|; J)^p_p &\le C \int_J \int_0^{|J|/\varphi(x)} \frac{\varphi(x)}{|J|} |\Delta^3_{h\varphi(x)}(f, x; J)|^p \, dh \, dx \\ &\le Cn \int_0^{C_2/n} \int_J |\Delta^3_{h\varphi(x)}(f, x; J)|^p \, dx \, dh. \end{aligned} \tag{3.17}$$

Note that in [DLY], (3.18) is only stated for $0 < p < 1$ but the proof is the same for all p. Using (3.16) and (3.17) in (3.15), we obtain

$$\|f - P_n\|^p_{L_p[-1,1]} \le Cn \int_0^{C_2/n} \int_{-1}^1 |\Delta^3_{h\varphi(x)}(f, x)|^p \, dx \, dh \le C\omega_3^\varphi(f, \frac{1}{n})^p_p$$

where we have used the fact that any $x \in [-1,1]$ appears in at most 12 of the intervals $J^*_{j,n}$. We have therefore proved Theorem 1.1.

4. Smoothing lemmas and the proof of Theorem 1.2. In Theorem 2.3 of §2, we have constructed a convex C^0 piecewise quadratic with good approximation properties. In this section, we shall use this piecewise quadratic and certain methods of smoothing to prove Theorem 1.2.

We begin with some simple results on spline functions. We use the notation in the book of DeVore and Lorentz [DLO]. Let $r > 0$ be an integer and let $\{z_i\}$ be a knot sequence on $\mathbb{R}$. This means that $z_j \le z_{j+1}$, and $z_j < z_{j+r}$ for all j. Associated to $\{z_j\}$, we have the B-splines

$$N_{j,r}(x) := N(x; z_j, \ldots, z_{j+r}) := (z_{j+r} - z_j)[z_j, \ldots, z_{j+r}](\cdot - x)^{r-1}_+.$$

We shall need the following result of de Boor (see DeVore and Lorentz [DLO, p.145]) on the stability of the B-spline basis.

Theorem 4.1. *There is a constant $D_r > 0$ such that for any spline $s = \sum_j c_j N_{j,r}$*

$$\begin{aligned} D_r \big\|\{d_j^{1/p} c_j\}\big\|_{\ell_p} &\le \|s\|_{L_p(\mathbb{R})} \le \big\|\{d_j^{1/p} c_j\}\big\|_{\ell_p}, && 1 \le p \le \infty \\ D_r \big\|\{d_j^{1/p} c_j\}\big\|_{\ell_p} &\le \|s\|_{L_p(\mathbb{R})} \le r^{1/p} \big\|\{d_j^{1/p} c_j\}\big\|_{\ell_p}, && 0 < p < 1. \end{aligned} \tag{4.1}$$

with $d_j := (z_{j+r} - z_j)/r$, for all j.

The following two lemmas will be used to eliminate discontinuities of piecewise polynomials.

Lemma 4.2. *Let $a < 0 < b$, and $h(x) := \beta x_+$ with $\beta \ge 0$. Then there exists a unique C^1 quadratic spline s on $[a, b]$ such that the only break point of s in (a, b) is a simple knot at 0 and*

$$s^{(j)}(a) = h^{(j)}(a), \quad s^{(j)}(b) = h^{(j)}(b), \qquad j = 0, 1, \tag{4.2}$$

and

$$s^{(j)}(x) \ge 0, \qquad x \in [a, b], \quad x \ne 0, \quad j = 0, 1, 2. \tag{4.3}$$

Moreover,

$$\|s - h\|_{L_p[a,b]} \le C\beta b^{1+1/p}, \tag{4.4}$$

and also

$$\|s - h\|_{L_p[a,b]} \le C \, \|h\|_{L_p[a,b]}, \tag{4.5}$$

where C depends only on the ratio $|a|/b$, and on p if $p \to 0$.

Proof. The uniqueness of the spline s is easy to prove and also follows from the Karlin-Ziegler theorem [DLO, p.162]. To prove the other properties of s, we denote by p_1 and p_2 the two quadratic pieces of s on $[a, 0]$ and $[0, b]$ respectively. Then

$$\begin{aligned} p_1(x) &:= -\frac{\beta b}{2a(b-a)}(x-a)^2 \\ p_2(x) &:= \beta x - \frac{\beta a}{2b(b-a)}(x-b)^2. \end{aligned}$$

It is trivial to verify

$$\begin{gathered} p_1(a) = p_1'(a) = 0, \qquad p_2(b) = \beta b, \qquad p_2'(b) = \beta \\ p_1(0) = p_2(0) = \frac{-\beta a b}{2(b-a)}, \qquad p_1'(0) = p_2'(0) = \frac{\beta b}{b-a} \\ p_1'' = \frac{-\beta b}{a(b-a)}, \qquad p_2'' = \frac{-\beta a}{b(b-a)}. \end{gathered}$$

This shows that p_1 and p_2 and their derivatives are nonnegative. Hence, we have (4.3).

To verify (4.4), we compute

$$\begin{aligned}\|s-h\|_{L_p[a,0]} &= \|p_1\|_{L_p[a,0]} \le |a|^{1/p}\,\|p_1\|_{L_\infty[a,0]} = |a|^{1/p}\frac{-\beta ab}{2(b-a)}\\ &\le |a|^{1/p}(\beta b) = (\frac{|a|}{b})^{1/p} b^{1/p}(\beta b) \le C\,\|h\|_{L_p[0,b]} = C\,\|h\|_{L_p[a,b]}\,,\end{aligned}$$

where we recall that the constant C depends on the ratio $|a|/b$ and on p as $p \to 0$. A similar proof yields

$$\|s-h\|_{L_p[0,b]} \le C|b|^{1/p}\beta b \le C\,\|h\|_{L_p[a,b]}\,.$$

Thus, we have verified (4.4) and (4.5). $\square$

Lemma 4.3. *Let $a<0<b$, $z_1=0$, $0<z_2<b$ and $h(x) := \gamma x_+^2$ with $\gamma \ge 0$. Then there exists a unique C^2 cubic spline s on $[a,b]$ such that the only breakpoints of s in (a,b) are two simple knots z_1 and z_2 and*

$$s^{(j)}(a) = h^{(j)}(a), \quad s^{(j)}(b) = h^{(j)}(b), \qquad j=0,1,2. \tag{4.6}$$

Moreover, we have

$$s^{(j)}(x) \ge 0, \qquad x\in[a,b], \quad j=0,1,2, \tag{4.7}$$

and

$$\|s-h\|_{L_p[a,b]} \le C\,\|h\|_{L_p[a,b]} \le C\gamma b^{2+1/p}, \tag{4.8}$$

where C depends on the ratio $|a|/b$, and also on p if $p\to 0$.

Proof. The uniqueness of the spline s again follows from the Karlin-Ziegler theorem. The proof below also gives the uniqueness. The first part of the proof is a modification of the proofs of Lemmas 6–8 of Y. K. Hu, D. Leviatan and X. M. Yu [HLY]; the reader can refer to that paper for more technical details. We are also influenced by Ivanov and Popov [IP].

We use auxiliary knots $z_{-3}=z_{-2}=z_{-1}=z_0 := a$ and $z_3=z_4=z_5=z_6 := b$. Then every cubic spline s with the knots $\{z_i\}_{i=-3}^{6}$ can be written as

$$s = \sum_{i=-3}^{2} c_{i,0}N_{i,4}.$$

Using the well-known differentiation formula for B-spline series (see, for example, [DLO, p.139])

$$\frac{d}{dx}(\sum_i c_iN_{i,r}) = (r-1)\sum_i \frac{c_i-c_{i-1}}{z_{i+r-1}-z_i}N_{i,r-1}, \tag{4.9}$$

we can express the conditions (4.6) as a system of equations in the $c_{i,0}$, which has a unique solution

$$\begin{cases} c_{-3,0} = c_{-2,0} = c_{-1,0} = 0 \\ c_{0,0} = \dfrac{\gamma}{3}(\delta_1^2 + \delta_1\delta_2) \\ c_{1,0} = \dfrac{\gamma}{3}(3\delta_1^2 + 4\delta_1\delta_2 + \delta_2^2) \\ c_{2,0} = \gamma(\delta_1 + \delta_2)^2, \end{cases}$$

where $\delta_j := z_{j+1} - z_j$, $j = 1, 2$. From this we compute the coefficients of

$$s' := \sum_{i=-2}^{2} c_{i,1} N_{i,3}$$

as

$$\begin{cases} c_{-2,1} = c_{-1,1} = 0 \\ c_{0,1} = \dfrac{(\delta_1^2 + \delta_1\delta_2)\gamma}{(\delta_0 + \delta_1 + \delta_2)} \\ c_{1,1} = \gamma(2\delta_1 + \delta_2) \\ c_{2,1} = 2\gamma(\delta_1 + \delta_2), \end{cases}$$

where $\delta_0 := z_1 - z_0$, and those of

$$s'' := \sum_{i=-1}^{2} c_{i,2} N_{i,2}$$

as

$$\begin{cases} c_{-1,2} = 0 \\ c_{0,2} = \dfrac{2\gamma(\delta_1^2 + \delta_1\delta_2)}{(\delta_0 + \delta_1)(\delta_0 + \delta_1 + \delta_2)} \\ c_{1,2} = \dfrac{2\gamma(2\delta_0\delta_1 + \delta_1^2 + \delta_0\delta_2 + 2\delta_1\delta_2 + \delta_2^2)}{(\delta_1 + \delta_2)(\delta_0 + \delta_1 + \delta_2)} \\ c_{2,2} = 2\gamma. \end{cases}$$

Since all $c_{i,j} \geq 0$, (4.7) follows from the positivity of the $N_{i,r}$.

We now verify (4.8). We have from the values of the $c_{j,0}$ and Theorem 4.1,

$$\|s\|_{L_p[a,b]} \leq C\gamma(\delta_1 + \delta_2)^2(|a| + b)^{1/p} \leq C\gamma b^{2+1/p} = C\|h\|_{L_p[0,b]} \tag{4.10}$$

where the constant depends on the ratio of $|a|$ and b. This implies (4.8). $\square$

The following theorem will prove the case of Theorem 1.2 dealing with C^1 piecewise quadratics. We use the notation of the introduction and §2. Given the partition $\mathbf{T}_n = \{t_j\}_{j=0}^n$, we let $z_j := t_{2j}$, $j = 0, \dots m$, with $m := [(n+1)/2]$ and let $\mathbf{Z}_n := \{z_j\}_{j=0}^m$. The intervals $I_j := [t_j, t_{j+1}]$ correspond to the partition $\mathbf{T}_n$ and the intervals $J_j := [z_j, z_{j+1}]$ correspond to the partition $\mathbf{Z}_n$. We recall our notation $I_j^{(1)} := [t_{j-11}, t_{j+11}]$.

Theorem 4.4. *For each partition $\mathbf{T}_n$, there is a convex, C^1 piecewise quadratic G on $\mathbf{T}_n$ that satisfies*

$$\|f-G\|_{L_p(I_j)} \le CE(I_j^{(1)}), \quad j=0,\dots,n-1, \tag{4.11}$$

where C depends on $\max_{0\le j\le n}(|I_{j\pm1}|/|I_j|)$ and on p as $p\to 0$.

Proof. Let g be the convex, C^0, piecewise quadratic of Theorem 2.3 for the partition $\mathbf{Z}_n$. We shall use Lemma 4.2 to add a knot t_{2j+1} on each interval $[z_j, z_{j+1}]$ and to create a C^1 piecewise quadratic G which has the desired properties. On each interval J_j, $g=g_j$ with g_j either r_j or r_{j+1}; the quadratic polynomials r_j are defined in §2 preceding Theorem 2.3.

We consider g on any interval $K_j := [t_{2j-1}, t_{2j+1}]$. Since g is continuous with only one breakpoint (at t_{2j}) on this interval, we have

$$g(x) = g_{j-1}(x) + \alpha_j (x-z_j)_+^2 + \beta_j (x-z_j)_+, \quad x\in K_j.$$

Since g is convex, we have $\beta_j \ge 0$. We apply Lemma 4.2 to obtain a C^1 piecewise quadratic s_j on K_j with a simple knot at z_j that approximates $\beta_j(x-z_j)_+$ as described in that lemma. We then define

$$G(x) = g_{j-1}(x) + \alpha_j(x-z_j)_+^2 + s_j(x), \quad x\in K_j, \quad j=1,\dots,n-1$$

and $G(x)=g(x)$ for all other $x\in[-1,1]$. Because of (4.2), G is in C^1 and is a piecewise quadratic on the partition $\mathbf{T}_n$. Because $g_{j-1}(x)+\alpha_j(x-z_j)_+^2$ has a nonnegative second derivative, property (4.3) guarantees that G is convex on each K_j. Because G is in C^1, it is convex on $[-1,1]$.

To complete the proof of the theorem, we need only show that

$$\|g-G\|_{L_p(K_j)} \le CE([z_{j-5}, z_{j+5}]). \tag{4.12}$$

According to (4.4), we have

$$\|g-G\|_{L_p(K_j)} = \|\beta_j(\cdot - z_j)_+ - s_j\|_{L_p(K_j)} \le C\beta_j |I_{2j}|^{1+1/p}. \tag{4.13}$$

We now estimate β_j in order to prove (4.12).

We have

$$\begin{aligned}
|\beta_j| = |g_j'(z_j) - g_{j-1}'(z_j)| &\le \left\|g_j' - g_{j-1}'\right\|_{L_\infty(J_{j-1})} \\
&\le C|z_j - z_{j-1}|^{-1} \left\|g_j - g_{j-1}\right\|_{L_\infty(J_{j-1})} \\
&\le C|J_{j-1}|^{-(1+1/p)} \|g_j - g_{j-1}\|_{L_p(J_{j-1})} \\
&\le C|J_{j-1}|^{-(1+1/p)} E([z_{j-5}, z_{j+5}])
\end{aligned}$$

where the last inequality is derived as in (2.4). Using this in (4.13) gives (4.12). $\square$

Remark. We have already mentioned in the introduction that Theorem 4.4 for a partition $\mathbf{T}_n$ of equidistant knots and approximation in the max-norm was proved by Ivanov and Popov [IP]. They inserted two additional knots between any two knots of the piecewise quadratic g. That it suffices to insert one knot between any two knots of g, was first pointed out to us by Dietrich Braess.

Finally, we consider the case of approximation by C^2 piecewise cubics. For the partition $\mathbf{T}_n := (t_j)$, we let $I_j^{(2)} := [t_{j-49}, t_{j+49}]$.

Theorem 4.5. *For any partition $\mathbf{T}_n$, there is a convex C^2 cubic spline s on $\mathbf{T}_n$, such that,*

$$\|f - s\|_{L_p(I_j)} \le CE(I_j^{(2)}), \quad j = 0, \ldots, n-1, \tag{4.14}$$

where C depends on $\max_{0 \le j \le n}(|I_{j\pm 1}|/|I_j|)$ and on p as $p \to 0$.

Proof. We define $\mathbf{Z}_n := \{z_j\}_{j=0}^m$ where $z_j := t_{4j}$, $j = 0, \ldots, m$, and $m := [(n+3)/4]$. According to Theorem 4.4 (for the partition $\mathbf{Z}_n$), there is a convex, C^1 piecewise quadratic G defined on $\mathbf{Z}_n$, which satisfies the conditions of that theorem.

We will use Lemma 4.3 to smooth G at the interior knots z_j, $j = 1, \ldots, m-1$. We let $G =: G_j$ on $J_j := [z_j, z_{j+1}]$. Then, each G_j is a convex quadratic polynomial and we have

$$G_j(x) - G_{j-1}(x) = \gamma_j (x - z_j)^2.$$

If $\gamma_j \ge 0$, then for $K_j := [t_{4j-2}, t_{4j+2}]$ and $j = 1, \ldots, m-1$, we have

$$G(x) = G_{j-1}(x) + \gamma_j (x - z_j)_+^2, \quad x \in K_j.$$

In this case, we let s_j be the C^2 piecewise cubic with simple knots at t_{4j}, t_{4j+1} given by Lemma 4.3 for the function $h_j(x) = \gamma_j (x - z_j)_+^2$ and $a = t_{4j-2}$, $b = t_{4j+2}$. We define

$$s(x) = G_{j-1}(x) + s_j(x), \quad x \in K_j.$$

Note that $s_j(x)$ has its simple knots at t_{4j}, t_{4j+1} and matches $\gamma_j(x - z_j)_+^2$ in value and first and second derivatives at the points t_{4j-2} and t_{4j+2}. Note also that in the case $j = m-1$, the points t_{4j+1} and t_{4j+2} may both equal one. In this case we set $s_{m-1} := 0$.

If $\gamma_j < 0$, we have

$$G(x) = G_j(x) - \gamma_j (z_j - x)_+^2, \quad x \in K_j.$$

In this case, we let s_j be the C^2 piecewise cubic of Lemma 4.3 with the simple knots at z_j and $2z_j - t_{4j-1}$ for the function $h_j(x) = -\gamma_j (x - z_j)_+^2$ and $a = 2z_j - t_{4j+2}$, $b = 2z_j - t_{4j-2}$. We define

$$s(x) := G_j(x) + s_j(2z_j - x), \quad x \in K_j.$$

Note that $s_j(2z_j - x)$ has its simple knots at t_{4j-1}, t_{4j} and matches $-\gamma_j(z_j - x)_+^2$ in value and first and second derivatives at the points t_{4j-2} and t_{4j+2}.

Because of (4.6), s is a C^2 piecewise cubic spline on the partition $\mathbf{T}_n$. Also, s_j is convex because of (4.7). Hence, s is also convex on $[-1, 1]$. On the interval K_j, we have from (4.11) and (4.8),

$$\begin{aligned} \|f - s\|_{L_p(K_j)} &\le \|f - G\|_{L_p(K_j)} + \|G - s\|_{L_p(K_j)} \\ &\le CE([z_{j-12}, z_{j+11}]) + C|\gamma_j||t_{4j+2} - t_{4j-2}|^{2+1/p}. \end{aligned} \tag{4.15}$$

To complete the proof, we need to estimate $|\gamma_j|$. We have

$$
\begin{aligned}
2|\gamma_j| = |G_j''(z_j) - G_{j-1}''(z_j)| \\
=& \|G_j'' - G_{j-1}''\|_{L_\infty(J_j)} \\
\le& C|J_j|^{-2}\|G_j - G_{j-1}\|_{L_\infty(J_j)} \\
\le& C|J_j|^{-2-1/p}\|G_j - G_{j-1}\|_{L_p(J_j)}.
\end{aligned}
\tag{4.16}
$$

We write $G_j - G_{j-1} = (f - G_{j-1}) - (f - G_j)$ to find

$$
\begin{aligned}
\|G_j - G_{j-1}\|_p &\le C\left(\|f - G_j\|_{L_p(J_j)} + \|f - G_{j-1}\|_{L_p(J_j)}\right) \\
&\le CE([z_{j-11}, z_{j+11}]) + E([z_{j-12}, z_{j+10}]),
\end{aligned}
$$

where we have used (4.11) and **F2**. If we use this last inequality in (4.16) and then in (4.15), we obtain

$$
\|f - s\|_{L_p(K_j)} \le CE([z_{j-12}, z_{j+11}]).
$$

Recalling that $z_j = t_{4j}$ and $K_j = [t_{4j-2}, t_{4j+2}]$, we see that we have proved (4.14). $\square$

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Department of Mathematics, University of South Carolina, Columbia, SC 29208

Department of Mathematics and Computer Science, Georgia Southern University, Statesboro, GA 30460-8093

Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel