

## Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$

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**Abstract:** A complete characterization is given of closed shift-invariant subspaces of  $L_2(\mathbb{R}^d)$  which provide a specified approximation order. When such a space is principal (i.e., generated by a single function), then this characterization is in terms of the Fourier transform of the generator. As a special case, we obtain the classical Strang-Fix conditions, but without requiring the generating function to decay at infinity. The approximation order of a general closed shift-invariant space is shown to be already realized by a specifiable principal subspace.

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## 1. Introduction

We are interested in the approximation properties of closed shift-invariant subspaces of  $L_2(\mathbb{R}^d)$ . We say that a space  $\mathcal{S}$  of complex-valued functions defined on  $\mathbb{R}^d$  is **shift-invariant** if, for each  $f \in \mathcal{S}$ , the space  $\mathcal{S}$  also contains the **shifts**  $f(\cdot + \alpha)$ ,  $\alpha \in \mathbb{Z}^d$ . In other words,  $\mathcal{S}$  contains all the *integer* translates of  $f$  if it contains  $f$ . A particularly simple example is provided by the space

$$\mathcal{S}_0(\phi)$$

of all *finite* linear combinations of shifts of a single function  $\phi$ . We call its  $L_2(\mathbb{R}^d)$ -closure the **principal** shift-invariant space **generated by**  $\phi$  and denote it by

$$\mathcal{S}(\phi).$$

Of course, a closed shift-invariant subspace of  $L_2(\mathbb{R}^d)$  need not be principal; it need not even be generated by the shifts of finitely many functions.

Shift-invariant spaces are important in a number of areas of analysis. Many spaces, encountered in approximation theory and in finite element analysis, are generated by the shifts of a finite number of functions  $\phi$  on  $\mathbb{R}^d$ . Shift-invariant spaces also play a key role in the construction of wavelets. In each of these applications, one is interested in how well a general function  $f$  can be approximated by the elements of the scaled spaces

$$\mathcal{S}^h := \{s(\cdot/h) : s \in \mathcal{S}\}.$$

We postpone discussion of the literature until we have introduced some additional terminology and stated our main results.

Associated to any closed subspace  $\mathcal{S}$  of  $L_2(\mathbb{R}^d)$  and any function  $f \in L_2(\mathbb{R}^d)$ , the approximation error is

$$(1.1) \quad E(f, \mathcal{S}) := \min\{\|f - s\| : s \in \mathcal{S}\}.$$

In this paper, we describe the properties of  $\mathcal{S}$  which govern the decay rates of  $E(f, \mathcal{S}^h)$ . We characterize when the scaled subspaces  $\mathcal{S}^h$  are **dense** in the sense that  $\lim_{h \rightarrow 0} E(f, \mathcal{S}^h) = 0$  for every  $f \in L_2(\mathbb{R}^d)$ . More generally, we characterize when the spaces  $\mathcal{S}^h$  approximate suitably smooth functions to order  $O(h^k)$  as  $h \rightarrow 0$ .

Our definitions of approximation orders are in terms of the potential space  $W_2^k(\mathbb{R}^d)$ ,  $k \in \mathbb{R}_+$ , defined by

$$W_2^k(\mathbb{R}^d) := \{f \in L_2(\mathbb{R}^d) : \|f\|_{W_2^k(\mathbb{R}^d)} := (2\pi)^{-d/2} \|(1 + |\cdot|)^k \widehat{f}\| < \infty\}.$$

(Here and later, we use  $|x| := (x_1^2 + \dots + x_d^2)^{1/2}$  to denote the Euclidean norm of a point  $x = (x_1, \dots, x_d)$  in  $\mathbb{R}^d$ .) When  $k$  is a positive integer, these are the usual Sobolev spaces. We say that  $\mathcal{S}$  **provides approximation order**  $k$  if, for every  $f \in W_2^k(\mathbb{R}^d)$ ,

$$(1.2) \quad E(f, \mathcal{S}^h) \leq \text{const} s h^k \|f\|_{W_2^k(\mathbb{R}^d)}.$$

A variant of this problem is to characterize when, for a given  $k \geq 0$ , we have for each  $f \in W_2^k(\mathbb{R}^d)$  (in addition to (1.2)),

$$(1.3) \quad E(f, \mathcal{S}^h) = o(h^k), \quad h \rightarrow 0.$$

When  $k = 0$ , this is the density problem. For this reason, we say that  $\mathcal{S}$  provides **density order**  $k$  whenever (1.3) holds.

Our characterizations of density, approximation order, and density order are in terms of Fourier transforms. If  $f \in L_1(\mathbb{R}^d)$ , its Fourier transform  $\hat{f}$  is defined by

$$\hat{f}(y) := \int_{\mathbb{R}^d} f(x)e^{-ix \cdot y} dx.$$

Many authors have shown (under various restrictive conditions on  $\phi$ ) that the approximation properties of a principal shift-invariant space  $\mathcal{S}(\phi)$  are related to *the order of the zeros of the Fourier transform* of  $\phi$  at the integer multiples of  $2\pi$ . It is therefore not surprising that our characterizations of approximation order involve the behavior near zero of the  $2\pi$ -periodization of  $|\hat{\phi}|^2$ , i.e., the  $L_2(\mathbb{T}^d)$ -function

$$(1.4) \quad [\hat{\phi}, \hat{\phi}] := \sum_{\beta \in 2\pi\mathbb{Z}^d} |\hat{\phi}(\cdot + \beta)|^2.$$

This function enters our considerations as part of the function  $\Lambda_\phi \in L_\infty(C)$ , defined on the centered cube in  $\mathbb{R}^d$  of side length  $2\pi$ :

$$(1.5) \quad \Lambda_\phi := \left(1 - \frac{|\hat{\phi}|^2}{[\hat{\phi}, \hat{\phi}]}\right)^{1/2}, \quad \text{on } C := [-\pi, \pi]^d.$$

Here (and below without further comment), we identify the space  $L_2(\mathbb{T}^d)$  of functions on the  $d$ -dimensional torus  $\mathbb{T}^d$  with the space  $L_2(C)$  of functions on the fundamental domain  $C$ . Our characterization of approximation order is in terms of the function  $y \mapsto \Lambda_\phi(y)/|y|^k$ .

It is the behaviour of  $\Lambda_\phi$  at the origin, or, more precisely, the behaviour of the function  $y \mapsto |y|^{-k}\Lambda_\phi(y)$ , that turns out to be crucial for the approximation order of  $\mathcal{S}(\varphi)$ . Indeed, we shall prove:

**Theorem 1.6.** *The principal shift-invariant subspace  $\mathcal{S}(\phi)$  of  $L_2(\mathbb{R}^d)$  provides approximation order  $k > 0$  if and only if  $|\cdot|^{-k}\Lambda_\phi$  is in  $L_\infty(C)$ .*

The analogue of this result for density orders is

**Theorem 1.7.** *The principal shift-invariant subspace  $\mathcal{S}(\phi)$  of  $L_2(\mathbb{R}^d)$  provides density order  $k \geq 0$  if and only if  $|\cdot|^{-k}\Lambda_\phi$  is in  $L_\infty(C)$  and*

$$(1.8) \quad \lim_{h \rightarrow 0} h^{-d} \int_{hC} |y|^{-2k} [\Lambda_\phi(y)]^2 dy = 0.$$

Of course, in the case  $k = 0$ , (1.8) characterizes when we have density.

It is rather remarkable that these conditions also characterize approximation and density orders for *arbitrary* closed shift-invariant subspaces of  $L_2(\mathbb{R}^d)$ . Namely, we shall prove:

**Theorem 1.9.** *A closed shift-invariant subspace  $\mathcal{S}$  of  $L_2(\mathbb{R}^d)$  provides approximation order  $k > 0$  if and only if it contains a function  $\phi$  for which  $|\cdot|^{-k}\Lambda_\phi$  is in  $L_\infty(C)$ . The space  $\mathcal{S}$  provides density order  $k \geq 0$  if and only if it contains a function  $\phi$  for which  $|\cdot|^{-k}\Lambda_\phi \in L_\infty(C)$  and (1.8) holds.*

We prove the last theorem by showing in §3 that the case of approximation by arbitrary closed shift-invariant subspaces of  $L_2(\mathbb{R}^d)$  can be reduced to the case of principal shift-invariant spaces.

In the case of principal shift-invariant spaces, our method of proof is based on two results which we feel will have other important applications. The first is an explicit formula for the best  $L_2(\mathbb{R}^d)$ -approximation from  $\mathcal{S}(\phi)$ . The second is the following characterization

$$(1.10) \quad \widehat{\mathcal{S}(\phi)} = \{\tau\hat{\phi} \in L_2(\mathbb{R}^d) : \tau \text{ is } 2\pi\text{-periodic}\}$$

of the space  $\mathcal{S}(\phi)$  in terms of its Fourier transform. Here and later, for a set  $F$  of functions, we denote by  $\widehat{F} := \{\widehat{f} : f \in F\}$  the set of its Fourier transforms.

It turns out that our analysis applies equally well to the more general situation where the  $h$ -refinement of the space  $\mathcal{S}$  is obtained by means other than scaling. Such cases are known and are of interest in both spline theory (e.g., exponential box splines, cf. [DR]) and radial basis function theory (cf. the detailed discussion in [BR2]). In the non-scaling case, we employ a family  $\{\mathcal{S}_h\}_h$  of shift-invariant spaces, and consider the rates of decay of  $E(f, \mathcal{S}_h^h)$  as a function of  $h$ . The notions of “approximation order  $k$ ” or “density order  $k$ ” for the sequence  $\{\mathcal{S}_h\}_h$  are obtained by replacing each  $E(f, \mathcal{S}^h)$  in the above definitions by  $E(f, \mathcal{S}_h^h)$ .

We close this section with a brief discussion of the connections between the results of this paper and results in the literature. Schoenberg, in his seminal paper [S], was the first to recognize the importance of the Fourier transform for describing approximation properties of principal shift-invariant spaces. For the case  $d = 1$ , and with  $\phi$  a piecewise continuous function with exponential decay at infinity, Schoenberg showed that all algebraic polynomials of degree  $< k$  can be written in the form  $\sum_{\alpha \in \mathbb{Z}^d} \phi(\cdot - \alpha)c(\alpha)$  in case

$$(1.11) \quad \widehat{\phi}(0) \neq 0 \quad \text{and} \quad D^\gamma \widehat{\phi} = 0 \quad \text{on} \quad 2\pi\mathbb{Z}^d \setminus 0 \quad \text{for all} \quad |\gamma| < k$$

holds (with  $d = 1$ ).

Strang and Fix [SF] have treated the approximation properties of the space

$$\mathcal{S}_*(\phi)$$

of all linear combinations  $\sum_{\alpha \in \mathbb{Z}^d} \phi(\cdot - \alpha)c(\alpha)$  (finite or not) of the integer shifts of a *compactly supported* function  $\phi$ . There is no problem of convergence of such sums since, for any point  $x \in \mathbb{R}^d$ , at most finitely many terms of the sum are nonzero at  $x$ . Strang and Fix necessarily restricted attention to the subspace

$$\mathcal{S}_2(\phi) := \mathcal{S}_*(\phi) \cap L_2(\mathbb{R}^d).$$

While this space is, in general, not closed in  $L_2(\mathbb{R}^d)$ , one can show (see Theorem 2.16 below) that its  $L_2(\mathbb{R}^d)$ -closure is  $\mathcal{S}(\phi)$ . Strang and Fix proved that  $\mathcal{S}_2(\phi)$  provides approximation order  $k$  whenever (1.11) holds.

To compare this result with Theorem 1.6 above, note that, for a compactly supported  $\phi$ ,  $[\widehat{\phi}, \widehat{\phi}]$  is a trigonometric polynomial, since then

$$(1.12) \quad [\widehat{\phi}, \widehat{\phi}] = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)e_\alpha, \quad \text{with} \quad a(\alpha) := \int_{\mathbb{R}^d} \phi(x - \alpha)\overline{\phi}(x) dx.$$

Here and later, we use the abbreviation

$$e_\alpha(y) := e^{i\alpha \cdot y}.$$

If (1.11) holds, then  $[\widehat{\phi}, \widehat{\phi}]$  does not vanish at the origin and  $\Lambda_\phi$  of (1.5) has a zero of multiplicity  $k$  there. Thus,  $|\cdot|^{-k}\Lambda_\phi$  is in  $L_\infty(C)$  (as we know it must be). However, there are two important points to bear in mind concerning our Theorem 1.6 and the Strang-Fix result. First of all, our theorem does not require that  $\phi$  be compactly supported, nor even that it decay at infinity. Secondly, it applies even when  $\widehat{\phi}$  vanishes at the origin, a case of practical importance yet not accessible to earlier approaches.

Actually, Strang and Fix proved more than we have just stated since they showed that the approximation order  $O(h^k)$  to a given  $f \in W_2^k(\mathbb{R}^d)$  by the elements of  $\mathcal{S}_2(\phi)^h$  can be achieved with a control on the coefficients of the approximants  $s_h \in \mathcal{S}_2(\phi)^h$ . Namely, if the approximants are represented with respect to the  $L_2$ -normalized functions  $\phi(\alpha, h, x) := h^{-d/2}\phi(x/h - \alpha)$  by  $s_h = \sum_{\alpha \in \mathbb{Z}^d} c_h(\alpha)\phi(\alpha, h, \cdot)$ , then

$$(1.13) \quad \|c_h\|_{\ell_2(\mathbb{Z}^d)} \leq \text{const}_f.$$

The introduction of such **controlled** approximation is important, since Strang and Fix show that, conversely, if  $\mathcal{S}_2(\phi)$  provides controlled approximation order  $k$ , then (1.11) holds. In other words, for compactly supported  $\phi$ ,  $\mathcal{S}_2(\phi)$  provides *controlled* approximation order  $k$  if and only if (1.11) holds. Since it can be easily seen that our condition in Theorem 1.6 is weaker than (1.11) (even for compactly supported  $\phi$ ), it follows that there are cases when the achievable approximation order cannot be obtained in a controlled manner. In this connection, it is worthwhile to point out (as is done in [SF]) that positive controlled approximation order forces  $\widehat{\phi}(0) \neq 0$ .

There is a rich literature of clarifications and extensions of the Strang-Fix result, including extensions to noncompactly supported  $\phi$  ([BH2], [J2], [DM2], [BJ], [B1], [R], [CL], [JL], [HL] [BR2]). In addition, there are many papers studying the approximation order of specific principal (and other) shift-invariant spaces, some of them ([Bu1,2], [BD], [BuD], [BH1], [BR1], [DJLR], [DM1], [DR], [Ja], [J1], [L], [LJ], [M], [MN1,2], [Ra], [RS]) are included in the references; see also the surveys [B2], [C], [P] and the references therein. By making assumptions on  $\phi$  weaker than those used in any of the above references, we can still translate our conditions on  $\Lambda_\phi$  into simple conditions on  $\widehat{\phi}$ . For example, we show in §5 the following:

**Theorem 1.14.** *Assume that  $\widehat{\phi}$  is bounded on some neighborhood of the origin. If  $\mathcal{S}(\phi)$  provides approximation order  $k$ , then  $\widehat{\phi}$  has a zero of order  $k$  at every  $\beta \in 2\pi\mathbb{Z}^d \setminus 0$ . In particular,  $D^\gamma \widehat{\phi}(\beta) = 0$  for all  $|\gamma| < k$  in case  $\widehat{\phi}$  is  $k$  times differentiable (in the classical sense) at such  $\beta$ .*

Note that the boundedness of  $\widehat{\phi}$  required here holds, for example, if  $\widehat{\phi}$  is continuous at 0. In particular, it holds for every  $\phi \in L_1(\mathbb{R}^d)$ .

We also show in §5 the following converse:

**Theorem 1.15.** *Assume that  $1/\widehat{\phi}$  is bounded on some neighborhood of the origin and that, for some  $\rho > k + d/2$ , all derivatives of  $\widehat{\phi}$  of order  $\leq \rho$  are in  $L_2(A)$ , with  $A := B_\varepsilon + (2\pi\mathbb{Z}^d \setminus 0)$  for some open ball  $B_\varepsilon$  centered at the origin. If  $D^\gamma \widehat{\phi}(\beta) = 0$  for all  $|\gamma| < k$  and all  $\beta \in 2\pi\mathbb{Z}^d \setminus 0$ , then  $\mathcal{S}(\phi)$  provides approximation order  $k$ .*

For most of the examples of a non-compactly supported  $\phi$  in the literature (e.g., radial basis functions, see [P]),  $\widehat{\phi}$  is very smooth on  $\mathbb{R}^d \setminus 0$ , but has a singularity at the origin. On the other hand, the present standard approach to the derivation of approximation orders (viz., the polynomial reproduction argument) requires  $\phi$  to decay at  $\infty$  (at least) like  $O(|\cdot|^{-(k+d)})$ , hence requires  $\widehat{\phi}$  to be *globally* smooth. To circumvent this obstacle, one usually seeks a function  $\psi \in \mathcal{S}_0(\phi)$  (or in some superspace of  $\mathcal{S}_0(\phi)$ ) whose Fourier transform  $\widehat{\psi}$  is smoother than  $\widehat{\phi}$ , since this implies a more favorable decay of  $\psi$  at  $\infty$ . This ‘localization’ process constitutes the main effort in establishing approximation orders for a non-compactly supported  $\phi$ . Our theorem, though, does not require  $\phi$  to decay at  $\infty$  at any particular rate, thus obviating the search for such  $\psi$ . Results (weaker than the above theorem) about  $L_\infty(\mathbb{R}^d)$ -approximation orders, that apply to functions which decay only mildly at  $\infty$ , were derived in [BR2]. The approach there exploits the fact that the exponential functions  $e_\theta$ ,  $\theta \in \mathbb{R}^d$ , are in the space in which approximation takes place. In contrast, the approach here makes use of the simple and explicit formula for the orthogonal projection onto  $\widehat{\mathcal{S}(\phi)}$ .

## 2. The orthogonal projector onto $\mathcal{S}(\phi)$

In this section, we derive two important facts about the principal shift-invariant space  $\mathcal{S}(\phi)$  which will be the basis of much of the analysis that follows. The first is a simple formula (given in Theorem 2.9) for the (Fourier transform of the) best  $L_2$ -approximation from  $\mathcal{S}(\phi)$ . The second is the description

$$(2.1) \quad \widehat{\mathcal{S}(\phi)} = \{\tau \widehat{\phi} \in L_2(\mathbb{R}^d) : \tau \text{ is } 2\pi\text{-periodic}\}$$

of  $\mathcal{S}(\phi)$  in terms of Fourier transforms mentioned in the introduction.

The yet to be proven (2.1) suggests that the calculation of integrals and inner products involving functions from  $\mathcal{S}(\phi)$  should be taken over the torus  $\mathbb{T}^d$ . This can be accomplished by periodization. If  $g \in L_1(\mathbb{R}^d)$ , then

$$(2.2) \quad \int_{\mathbb{R}^d} g = \sum_{\beta \in 2\pi\mathbb{Z}^d} \int_{C+\beta} g = \int_C g^\circ,$$

with

$$g^\circ := \sum_{\beta \in 2\pi\mathbb{Z}^d} g(\cdot + \beta)$$

the  $(2\pi)$ -**periodization** of  $g$ . Here, the sum is to be taken in the sense of  $L_1(\mathbb{T}^d)$ -convergence, which makes sense since, by assumption,  $g \in L_1(\mathbb{R}^d)$ . In particular,  $g^\circ \in L_1(\mathbb{T}^d)$ .

Similarly, we have

$$(2.3) \quad \int_{\mathbb{R}^d} g_0 \bar{g}_1 = \int_C [g_0, g_1]$$

for the inner product of two functions  $g_0, g_1 \in L_2(\mathbb{R}^d)$ , with

$$(2.4) \quad [g_0, g_1] := (g_0 \bar{g}_1)^\circ = \sum_{\beta \in 2\pi\mathbb{Z}^d} g_0(\cdot + \beta) \bar{g}_1(\cdot + \beta).$$

Note that  $[g_0, g_1]$  is in  $L_1(\mathbb{T}^d)$  since  $g_0 \bar{g}_1 \in L_1(\mathbb{R}^d)$ . Also, by the Cauchy-Schwarz inequality,

$$(2.5) \quad |[g_0, g_1]|^2 \leq [g_0, g_0][g_1, g_1],$$

and the right side of (2.5) is finite a.e. We will most often use (2.3) in the form

$$(2.6) \quad \int_{\mathbb{R}^d} \tau f \bar{\phi} = \int_C \tau [f, \hat{\phi}]$$

which is valid for arbitrary  $f, \phi \in L_2(\mathbb{R}^d)$  and arbitrary  $2\pi$ -periodic  $\tau$  for which  $\tau \hat{f} \in L_2(\mathbb{R}^d)$ . We note that (2.6) implies the estimate

$$(2.7) \quad \|\tau \hat{\phi}\|_{L_2(\mathbb{R}^d)} \leq \|\tau\|_{L_2(\mathbb{T}^d)} \|\hat{\phi}, \hat{\phi}\|_{L_\infty(\mathbb{T}^d)}^{1/2}$$

of use when  $[\hat{\phi}, \hat{\phi}]$  is bounded, e.g., when  $\phi$  is compactly supported.

After these brief remarks, let us consider the problem of finding a formula for the projection of  $L_2(\mathbb{R}^d)$  onto  $\mathcal{S}(\phi)$ . Let  $P := P_\phi$  denote the orthogonal projector onto  $\mathcal{S}(\phi)$ . Then  $Pf$  is the unique best  $L_2(\mathbb{R}^d)$ -approximation to  $f$  from  $\mathcal{S}(\phi)$ , and is characterized by the fact that it lies in  $\mathcal{S}(\phi)$  while its difference from  $f$  is orthogonal to  $\mathcal{S}(\phi)$ . Since the Fourier transform preserves orthogonality, it follows (for example from the uniqueness of best approximation in  $L_2(\mathbb{R}^d)$ ) that the orthogonal projector  $\hat{P}$  onto  $\widehat{\mathcal{S}(\phi)}$  satisfies  $\hat{P}\hat{f} = \widehat{Pf}$ .

We consider first what it means for a function  $f$  to be orthogonal to  $\mathcal{S}(\phi)$ . Since finite linear combinations of the (integer) shifts  $\phi(\cdot + \alpha)$  of  $\phi$  are dense in  $\mathcal{S}(\phi)$ ,  $f \in L_2(\mathbb{R}^d)$  is orthogonal to  $\mathcal{S}(\phi)$  iff  $\hat{f}$  is orthogonal to  $e_{-\alpha}\hat{\phi}$  for every  $\alpha \in \mathbb{Z}^d$ , i.e. (with (2.6)), iff

$$0 = \int_{\mathbb{R}^d} \hat{f} e_\alpha \bar{\phi} = \int_C [\hat{f}, \hat{\phi}] e_\alpha \quad \text{for all } \alpha \in \mathbb{Z}^d.$$

This proves

**Lemma 2.8.** *The orthogonal complement  $\mathcal{S}(\phi)^-$  of  $\mathcal{S}(\phi)$  in  $L_2(\mathbb{R}^d)$  consists of exactly those  $f \in L_2(\mathbb{R}^d)$  for which  $[\widehat{f}, \widehat{\phi}] = 0$ .*

From Lemma 2.8, we can easily determine  $Pf$ . Suppose, as is suggested by (2.1), that  $\widehat{Pf} = \tau\widehat{\phi}$ , with  $\tau$  some  $2\pi$ -periodic function. Then, from Lemma 2.8,

$$[\widehat{f}, \widehat{\phi}] = [\widehat{Pf}, \widehat{\phi}] = [\tau\widehat{\phi}, \widehat{\phi}] = \tau[\widehat{\phi}, \widehat{\phi}].$$

This motivates the following:

**Theorem 2.9.** *For each  $f \in L_2(\mathbb{R}^d)$ ,  $\widehat{P_\phi}f = \tau_f\widehat{\phi}$ , with the  $2\pi$ -periodic function  $\tau_f$  defined by*

$$(2.10) \quad \tau_f := \begin{cases} [\widehat{f}, \widehat{\phi}]/[\widehat{\phi}, \widehat{\phi}] & \text{on } \Omega_\phi; \\ 0 & \text{otherwise,} \end{cases}$$

and  $\Omega_\phi$  defined up to a null-set by

$$\Omega_\phi := \text{supp}[\widehat{\phi}, \widehat{\phi}] := \{\omega \in \mathbb{T}^d : [\widehat{\phi}, \widehat{\phi}](\omega) \neq 0\}.$$

**Proof:** It is enough to show that  $\widehat{Pf} = \tau_f\widehat{\phi}$  for each  $f \in L_2(\mathbb{R}^d)$ . We first want to see that  $\tau_f\widehat{\phi}$  is in  $L_2(\mathbb{R}^d)$ . By (2.5),  $|\tau_f|^2[\widehat{\phi}, \widehat{\phi}] \leq [\widehat{f}, \widehat{f}]$ . With this, two applications of (2.6) give

$$(2.11) \quad \int_{\mathbb{R}^d} |\tau_f\widehat{\phi}|^2 = \int_C |\tau_f|^2[\widehat{\phi}, \widehat{\phi}] \leq \int_C [\widehat{f}, \widehat{f}] = \int_{\mathbb{R}^d} |\widehat{f}|^2.$$

Consequently,  $\tau_f\widehat{\phi} \in L_2(\mathbb{R}^d)$  and moreover the linear map

$$Q : L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d) : f \mapsto \tau_f\widehat{\phi}$$

is well-defined and norm-reducing on  $L_2(\mathbb{R}^d)$ . We next prove that  $Q = \widehat{P}$ .

If  $\widehat{f} \in \widehat{\mathcal{S}(\phi)^-} = (\mathcal{S}(\phi)^-)^{\widehat{\cdot}}$ , then Lemma 2.8 gives that  $\tau_f = 0$ , hence  $Q\widehat{f} = 0$ . Thus  $Q = \widehat{P}$  on  $\widehat{\mathcal{S}(\phi)^-}$ . On the other hand, on  $\Omega_\phi = \text{supp}[\widehat{\phi}, \widehat{\phi}]$ ,

$$\tau_{\phi(\cdot+\alpha)} = [e_\alpha\widehat{\phi}, \widehat{\phi}]/[\widehat{\phi}, \widehat{\phi}] = e_\alpha, \quad \text{for all } \alpha \in \mathbb{Z}^d.$$

Since  $\widehat{\phi} = 0$  on the complement of  $\Omega_\phi + 2\pi\mathbb{Z}^d$ , this implies that  $Q$  maps the Fourier transform of each integer shift of  $\phi$  to itself. Since  $Q$  is linear and bounded, and coincides with  $\widehat{P}$  on a fundamental set for  $\widehat{\mathcal{S}(\phi)^-}$ , we have  $Q = \widehat{P}$  on  $\widehat{\mathcal{S}(\phi)^-}$ . By linearity,  $Q = \widehat{P}$  on all of  $L_2(\mathbb{R}^d)$ .  $\square$

**Remark.** With the convention (which we use throughout this paper) that 0 times any extended number is 0, we are entitled to write

$$(2.12) \quad \tau_f = [\widehat{f}, \widehat{\phi}]/[\widehat{\phi}, \widehat{\phi}] \quad \text{and} \quad \widehat{P_\phi}f = \frac{[\widehat{f}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]} \widehat{\phi}.$$

Note that (2.11) supplies the following lemma.

**Lemma 2.13.** *If  $\phi, f \in L_2(\mathbb{R}^d)$ , then  $\tau_f\widehat{\phi} \in L_2(\mathbb{R}^d)$ , and  $\|\tau_f\widehat{\phi}\| \leq \|\widehat{f}\|$ .*

As a consequence, we obtain the characterization (2.1) of the space  $\mathcal{S}(\phi)$  in terms of its Fourier transform.

**Theorem 2.14.** *A function  $f$  is in  $\mathcal{S}(\phi)$  if and only if  $\widehat{f} = \tau\widehat{\phi}$  for some  $2\pi$ -periodic  $\tau$  with  $\tau\widehat{\phi} \in L_2(\mathbb{R}^d)$ . In particular,  $\tau\widehat{\phi} \in \widehat{\mathcal{S}(\phi)}$  for every bounded  $\tau$ .*

**Proof:** If  $f \in \mathcal{S}(\phi)$ , then  $Pf = f$ . Hence, by Theorem 2.9,  $\widehat{f} = \tau_f\widehat{\phi}$  with  $\tau_f$  the  $2\pi$ -periodic function  $[\widehat{f}, \widehat{\phi}]/[\widehat{\phi}, \widehat{\phi}]$ , and  $\tau_f\widehat{\phi} \in L_2(\mathbb{R}^d)$  because of Lemma 2.13.

Conversely, if  $\tau$  is defined on  $\mathbb{T}^d$ , and  $\tau\widehat{\phi} \in L_2(\mathbb{R}^d)$ , then the inverse transform  $f$  of  $\tau\widehat{\phi}$  is also in  $L_2(\mathbb{R}^d)$  and satisfies  $\tau_f = [\tau\widehat{\phi}, \widehat{\phi}]/[\widehat{\phi}, \widehat{\phi}] = \tau$  on  $\Omega_\phi = \text{supp}[\widehat{\phi}, \widehat{\phi}]$ . Since  $\widehat{\phi}$  vanishes off  $\Omega_\phi + 2\pi\mathbb{Z}^d$ , this implies with Theorem 2.9 that

$$\widehat{Pf} = \tau_f\widehat{\phi} = \tau\widehat{\phi} = \widehat{f}.$$

Consequently,  $Pf = f$  and hence  $f \in \mathcal{S}(\phi)$ . Finally, if  $\tau$  is bounded, then  $\tau\widehat{\phi} \in L_2(\mathbb{R}^d)$  since  $\widehat{\phi} \in L_2(\mathbb{R}^d)$ .  $\square$

**Remark 2.15.** Asher Ben-Artzi has pointed out to us that Theorem 2.14 could have been derived from general results (cf. Theorem 8 of [H;p.59]) concerning closed subspaces of  $L_2(\mathbb{T}, \ell_2)$  which are invariant under multiplication by exponentials. Furthermore, the lemma of [H;p.58] shows that Theorem 2.14 implies Theorem 2.9.

**Remark.** The representation  $\tau\widehat{\phi}$  for  $\widehat{f} \in \widehat{\mathcal{S}(\phi)}$  is in general not unique. If  $\tau_0\widehat{\phi} = \tau_1\widehat{\phi}$ , we can only conclude that  $\tau_0 = \tau_1$  a.e. on  $\Omega_\phi$ . However, if the shifts of  $\phi$  are an orthonormal basis or, more generally, an  $L_2(\mathbb{R}^d)$ -stable basis, then, as is well known,  $[\widehat{\phi}, \widehat{\phi}]$  and its reciprocal are both in  $L_\infty$  and not only is the representation unique but the function  $\tau$  is in  $L_2(\mathbb{T}^d)$ . It is interesting to note further that we have a unique representation even when the shifts of  $\phi$  are not an  $L_2(\mathbb{R}^d)$ -stable basis provided  $\Omega_\phi$  differs from  $\mathbb{T}^d$  only by a null-set.

The following consequence of Theorem 2.14 is of importance when comparing our results with related results in the literature.

**Theorem 2.16.** *If  $\phi \in L_2(\mathbb{R}^d)$  has compact support, then  $\mathcal{S}(\phi)$  is the  $L_2(\mathbb{R}^d)$ -closure of  $\mathcal{S}_2(\phi) = \mathcal{S}_*(\phi) \cap L_2(\mathbb{R}^d)$ .*

**Proof:** Since  $\mathcal{S}(\phi)$  is the  $L_2(\mathbb{R}^d)$ -closure of  $\mathcal{S}_0(\phi)$  and  $\mathcal{S}_0(\phi)$  is contained in  $\mathcal{S}_2(\phi)$  (since  $\phi \in L_2(\mathbb{R}^d)$ ), we only have to prove that

$$(2.17) \quad \mathcal{S}_2(\phi) \subset \mathcal{S}(\phi).$$

We now prove this by showing that  $P_\phi f = f$  for every  $f \in \mathcal{S}_2(\phi)$ , i.e., with (2.12), that

$$(2.18) \quad \widehat{f} = \frac{[\widehat{f}, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]} \widehat{\phi}.$$

Since  $\phi$  has compact support,  $[\widehat{\phi}, \widehat{\phi}]$  is a trigonometric polynomial (cf. (1.12)), hence (2.18) is equivalent to the equation

$$(2.19) \quad [\widehat{\phi}, \widehat{\phi}]\widehat{f} = [\widehat{f}, \widehat{\phi}]\widehat{\phi} \quad a.e.,$$

and it is this equation we now verify for any  $f$  in  $L_2(\mathbb{R}^d)$  of the form  $\sum_{\beta \in \mathbb{Z}^d} \phi(\cdot - \beta)c(\beta)$ .

We do this by showing that both sides of (2.19) are the Fourier transform of the function

$$\sum_{\alpha \in \mathbb{Z}^d} f(\cdot + \alpha)a(\alpha),$$

with  $a(\alpha) = \int_{\mathbb{R}^d} \phi(\cdot - \alpha) \overline{\phi}$  the (Fourier) coefficients of the trigonometric polynomial  $[\widehat{\phi}, \widehat{\phi}]$ , see (1.12). This is immediate for the left side of (2.19) since  $(\sum_{\alpha \in \mathbb{Z}^d} f(\cdot + \alpha) a(\alpha))^\wedge = (\sum_{\alpha \in \mathbb{Z}^d} a(\alpha) e_\alpha)^\wedge f$  for any  $f \in L_2(\mathbb{R}^d)$  and any finite sequence  $(a(\alpha))$ , and  $[\widehat{\phi}, \widehat{\phi}]$  is indeed a finite sum of exponentials since  $\phi$  is compactly supported. As to the right side of (2.19),  $[\widehat{f}, \widehat{\phi}]$  is a  $2\pi$ -periodic  $L_2$ -function (since  $\phi$  is compactly supported, thus  $\widehat{\phi}$  is bounded), hence the  $L_2(\mathbb{T}^d)$ -limit of its Fourier series  $\sum_{\gamma \in \mathbb{Z}^d} b(\gamma) e_\gamma$ , with  $b$  given by

$$b(\gamma) := (2\pi)^{-d} \int_C [\widehat{f}, \widehat{\phi}] e_{-\gamma} = \int_{\mathbb{R}^d} f \overline{\phi}(\cdot + \gamma) = \sum_{\beta \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \phi(\cdot - \beta) c(\beta) \overline{\phi}(\cdot + \gamma) = \sum_{\beta \in \mathbb{Z}^d} c(\beta) a(\gamma + \beta).$$

By (2.7),  $[\widehat{f}, \widehat{\phi}] \widehat{\phi}$  is the  $L_2(\mathbb{R}^d)$ -limit of  $\sum_{\gamma \in \mathbb{Z}^d} b(\gamma) e_\gamma \widehat{\phi}$ , hence  $([\widehat{f}, \widehat{\phi}] \widehat{\phi})^\vee$  is the  $L_2(\mathbb{R}^d)$ -limit of  $\sum_{\gamma \in \mathbb{Z}^d} \phi(\cdot + \gamma) b(\gamma)$ . Since this last sum also converges uniformly on compact sets, these two limits must be the same. This implies that the right side of (2.19) is the Fourier transform of

$$\sum_{\gamma \in \mathbb{Z}^d} \phi(\cdot + \gamma) \sum_{\beta \in \mathbb{Z}^d} c(\beta) a(\gamma + \beta) = \sum_{\alpha \in \mathbb{Z}^d} \sum_{\beta \in \mathbb{Z}^d} \phi(\cdot + \alpha - \beta) c(\beta) a(\alpha) = \sum_{\alpha \in \mathbb{Z}^d} f(\cdot + \alpha) a(\alpha),$$

with the rearrangement of the sums justified by the fact that all sums are finite.  $\square$

We now turn to our main objective, viz. the error of the best approximation. If  $\widehat{f}$  is supported in one the cubes  $\beta + C$ ,  $\beta \in 2\pi\mathbb{Z}^d$ , this error takes a very simple form:

**Theorem 2.20.** *Let  $\phi \in L_2(\mathbb{R}^d)$ . If  $f \in L_2(\mathbb{R}^d)$  and  $\text{supp } \widehat{f} \subset \beta + C$  for some  $\beta \in 2\pi\mathbb{Z}^d$ , then*

$$(2.21) \quad E(f, \mathcal{S}(\phi))^2 = (2\pi)^{-d} E(\widehat{f}, \widehat{\mathcal{S}(\phi)})^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} |\widehat{f}|^2 \left(1 - \frac{|\widehat{\phi}|^2}{[\widehat{\phi}, \widehat{\phi}]}\right).$$

**Proof:** Since  $\text{supp } \widehat{f} \subset C + \beta$  for some  $\beta \in 2\pi\mathbb{Z}^d$ , we have  $[\widehat{f}, \widehat{\phi}] = \widehat{f}(\cdot + \beta) \overline{\widehat{\phi}(\cdot + \beta)}$  on  $C$ . Therefore, with (2.6),

$$\|\tau_f \widehat{\phi}\|^2 = \int_C |\widehat{f}(\cdot + \beta)|^2 |\widehat{\phi}(\cdot + \beta)|^2 / [\widehat{\phi}, \widehat{\phi}] = \int_{\mathbb{R}^d} |\widehat{f}|^2 |\widehat{\phi}|^2 / [\widehat{\phi}, \widehat{\phi}].$$

By Theorem 2.9, this shows that

$$\|P_\phi f\|^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} |\widehat{f}|^2 |\widehat{\phi}|^2 / [\widehat{\phi}, \widehat{\phi}],$$

and this finishes the proof since  $\|f - P_\phi f\|^2 = \|f\|^2 - \|P_\phi f\|^2$ .  $\square$

### 3. The reduction to the principal case

The explicit and simple expression, derived in the previous section, for the orthogonal projector onto a *principal* shift-invariant space will also prove to be very useful in the discussion of approximation from a *general* shift-invariant space. For, remarkably, the approximation power of a general shift-invariant space, however large, is already contained in a single (suitably chosen) principal shift-invariant subspace of it. The next proposition provides the algebraic background for this fact. We use repeatedly the simple observation that the best approximation  $Pf$  to  $f$  from  $\mathcal{S}$  is also the best approximation  $P_{P_f} f$  to  $f$  from  $\mathcal{S}(Pf)$ , i.e.,

$$P_{P_f} f = Pf.$$

**Proposition 3.1.** *Let  $P$  be the orthogonal projector onto the closed shift-invariant subspace  $\mathcal{S}$  of  $L_2(\mathbb{R}^d)$  and denote by  $\widehat{P}$  the corresponding orthogonal projector onto  $\widehat{\mathcal{S}}$ . Then  $\widehat{P}(\tau\widehat{f}) = \tau\widehat{P}(\widehat{f})$  for any  $f \in L_2(\mathbb{R}^d)$  and any  $2\pi$ -periodic  $\tau$  for which  $\tau\widehat{f} \in L_2(\mathbb{R}^d)$ .*

**Proof:** If  $\mathcal{S}$  is principal, then the conclusion follows directly from (2.12). For the general case, the assumptions on  $\tau$  and  $\widehat{f}$  imply with Theorem 2.14 that  $\tau\widehat{f} \in \widehat{\mathcal{S}}(\widehat{f})$ . Since  $\mathcal{S}(f)$  is, by definition, the  $L_2(\mathbb{R}^d)$ -closure of  $\mathcal{S}_0(f)$ , and  $\widehat{\mathcal{S}}_0(\widehat{f}) = \{\tau_n\widehat{f} : \tau_n \text{ a trig.polynomial}\}$ , it follows that  $\tau\widehat{f}$  is the  $L_2(\mathbb{R}^d)$ -limit of  $\tau_n\widehat{f}$  for some sequence  $(\tau_n)$  of trigonometric polynomials. The shift-invariance of  $\mathcal{S}$  and the uniqueness of the best  $L_2$ -approximation imply that  $P(f(\cdot + \alpha)) = (Pf)(\cdot + \alpha)$  for every  $f \in L_2(\mathbb{R}^d)$  and every  $\alpha \in \mathbb{Z}^d$ . Hence, taking finite linear combinations of Fourier transforms,  $\widehat{P}(\tau_n\widehat{f}) = \tau_n\widehat{P}f$ , and so, by the continuity of  $\widehat{P}$ ,

$$\widehat{P}(\tau\widehat{f}) = \lim_{n \rightarrow \infty} \widehat{P}(\tau_n\widehat{f}) = \lim_{n \rightarrow \infty} \tau_n\widehat{P}f.$$

Each  $\tau_n\widehat{P}f$  is in the closed space  $\widehat{\mathcal{S}}(\widehat{P}f)$ , therefore also  $\widehat{P}(\tau\widehat{f})$  lies in  $\widehat{\mathcal{S}}(\widehat{P}f)$ . Thus, projecting  $\tau\widehat{f}$  onto  $\widehat{\mathcal{S}}$  is the same as projecting it onto the subspace  $\widehat{\mathcal{S}}(\widehat{P}f)$  of  $\widehat{\mathcal{S}}$ . Since we already know that  $\widehat{P}_\phi(\tau\widehat{f}) = \tau\widehat{P}_\phi f$  for any  $\phi, f \in L_2(\mathbb{R}^d)$ , this means that we obtain

$$\widehat{P}(\tau\widehat{f}) = \widehat{P}_{Pf}(\tau\widehat{f}) = \tau\widehat{P}_{Pf}(\widehat{f}) = \tau\widehat{P}f,$$

the last equality since  $P_{Pf}f = Pf$ . □

**Corollary 3.2.** *If  $P$  is the orthogonal projector onto some shift-invariant subspace of  $L_2(\mathbb{R}^d)$  and  $g \in L_2(\mathbb{R}^d)$ , then*

$$PP_g = P_{P_g}P_g.$$

**Proof:** If  $f \in L_2(\mathbb{R}^d)$ , then  $\widehat{P}_g f = \tau\widehat{g}$  for some  $2\pi$ -periodic  $\tau$  and therefore by Proposition 3.1,  $\widehat{P}(\widehat{P}_g f) = \tau\widehat{P}g$ . On the other hand,  $\widehat{P}_{P_g}(\widehat{P}_g f) = \widehat{P}_{P_g}(\tau\widehat{g}) = \tau\widehat{P}_{P_g}\widehat{g} = \tau\widehat{P}g$ . □

**Theorem 3.3.** *For any closed shift-invariant subspace  $\mathcal{S}$  of  $L_2(\mathbb{R}^d)$  and any  $f, g \in L_2(\mathbb{R}^d)$ ,*

$$(3.4) \quad E(f, \mathcal{S}) \leq E(f, \mathcal{S}(Pg)) \leq E(f, \mathcal{S}) + 2E(f, \mathcal{S}(g)),$$

with  $P = P_{\mathcal{S}}$  the orthogonal projector onto  $\mathcal{S}$ .

**Proof:** Only the second inequality needs proof. By Corollary 3.2,

$$f - P_{P_g}f = f - Pf + Pf - PP_gf + P_{P_g}P_gf - P_{P_g}f,$$

and therefore

$$(3.5) \quad \|f - P_{P_g}f\| \leq \|f - Pf\| + \|f - P_gf\| + \|P_gf - f\|.$$

□

This theorem shows that the approximation order of the particular principal subspace  $\mathcal{S}(Pg)$  of  $\mathcal{S}$  is the same as that of all of  $\mathcal{S}$ , provided that the approximation order of the principal space  $\mathcal{S}(g)$  is at least as good as that of  $\mathcal{S}$ . This suggests the use of a special function  $g^*$  for which  $\mathcal{S}(g^*)$  has arbitrarily high approximation order. We can take  $g^*$  to be the inverse Fourier transform of the characteristic function of the cube  $C = [-\pi \dots \pi]^d$ , i.e.,

$$g^* := (\chi_C)^\vee.$$

We note that, by (2.12),  $\widehat{P_{g^*} f} = [\widehat{f}, \chi_C] / [\chi_C, \chi_C] \chi_C = \chi_C \widehat{f}$ . Hence,

$$(3.6) \quad E(f, \mathcal{S}(g^*)) = (2\pi)^{-d/2} \|(1 - \chi_C) \widehat{f}\|.$$

This allows us to show easily that the space  $\mathcal{S}(g^*)$  provides approximation and density order  $k$  for all  $k \geq 0$ . For this, we follow the example of [BR2] and consider, equivalently, the approximation of the scaled function

$$f_h := f(h \cdot)$$

from the fixed space  $\mathcal{S}$  instead of the approximation of the function  $f$  from the scaled space  $\mathcal{S}^h$ . For,

$$(3.7) \quad E(f, \mathcal{S}^h) = h^{d/2} E(f_h, \mathcal{S}),$$

as is easily established by a change of variables.

**Lemma 3.8.** *Let  $f \in W_2^k(\mathbb{R}^d)$ ,  $k \geq 0$ ,  $h > 0$ . Then*

$$E(f, \mathcal{S}(g^*)^h) \leq \varepsilon_f(h) h^k \|f\|_{W_2^k(\mathbb{R}^d)},$$

with the (nonnegative) function  $\varepsilon_f$  defined by

$$(3.9) \quad \varepsilon_f(h)^2 := \frac{\int_{(\mathbb{R}^d \setminus C)/h} (1 + |\cdot|)^{2k} |\widehat{f}|^2}{\int_{\mathbb{R}^d} (1 + |\cdot|)^{2k} |\widehat{f}|^2},$$

hence  $\varepsilon_f(h) \leq 1$ , and  $\varepsilon_f(0+) = 0$ .

**Proof:** Since  $f \in W_2^k(\mathbb{R}^d)$ , the function  $\nu := (1 + |\cdot|)^k \widehat{f}$  is in  $L_2(\mathbb{R}^d)$ , and  $\|f\|_{W_2^k(\mathbb{R}^d)} = (2\pi)^{-d/2} \|\nu\|$ . Since  $\widehat{f}_h = h^{-d} \widehat{f}(\cdot/h)$ , (3.7) and (3.6) imply that

$$\begin{aligned} (2\pi)^d E(f, \mathcal{S}(g^*)^h)^2 &= (2\pi)^d h^d E(f_h, \mathcal{S}(g^*))^2 = h^d \|(1 - \chi_C) \widehat{f}_h\|^2 = h^d \int_{\mathbb{R}^d \setminus C} |\widehat{f}_h(y)|^2 dy \\ &= h^{-d} \int_{\mathbb{R}^d \setminus C} |\widehat{f}(y/h)|^2 dy = h^{2k-d} \int_{\mathbb{R}^d \setminus C} \frac{|\nu(y/h)|^2}{(h + |y|)^{2k}} dy \\ &\leq h^{2k-d} \int_{\mathbb{R}^d \setminus C} |\nu(y/h)|^2 dy = h^{2k} \int_{(\mathbb{R}^d \setminus C)/h} |\nu|^2 = (2\pi)^d h^{2k} \varepsilon_f(h)^2 \|f\|_{W_2^k(\mathbb{R}^d)}^2. \end{aligned}$$

□

We note for later reference the following useful result established during the proof of Lemma 3.8.

**Corollary 3.10.** *For each  $f \in W_2^k(\mathbb{R}^d)$ ,*

$$h^{d/2} \|(1 - \chi_C) \widehat{f}_h\| \leq (2\pi)^{d/2} \varepsilon_f(h) h^k \|f\|_{W_2^k(\mathbb{R}^d)},$$

with  $\varepsilon_f$  given by (3.9).

Now let  $\mathcal{S}$  be an arbitrary closed shift-invariant subspace of  $L_2(\mathbb{R}^d)$  and let  $\phi^* := Pg^*$  be the best  $L_2(\mathbb{R}^d)$ -approximation to  $g^*$  from  $\mathcal{S}$ . Using (3.7) and Lemma 3.8 in (3.4), we obtain

$$(3.11) \quad E(f, \mathcal{S}^h) \leq E(f, \mathcal{S}(\phi^*)^h) \leq E(f, \mathcal{S}^h) + 2\varepsilon_f(h) h^k \|f\|_{W_2^k(\mathbb{R}^d)},$$

with  $\varepsilon_f(h)$  given by (3.9). This means that  $\mathcal{S}$  provides approximation order  $k > 0$  or density order  $k \geq 0$  if and only if its principal shift-invariant subspace  $\mathcal{S}(\phi^*)$  does. More than that, since  $\varepsilon_f(h)$  does not depend on  $\mathcal{S}$ , it proves the following:

**Theorem 3.12.** *The sequence  $\{\mathcal{S}_h\}_h$  of closed shift-invariant subspaces of  $L_2(\mathbb{R}^d)$  provides approximation order  $k > 0$  or density order  $k \geq 0$  if and only if the corresponding sequence  $\{\mathcal{S}(\phi_h^*)\}_h$  of principal shift-invariant subspaces (with  $\phi_h^* := P_{\mathcal{S}_h}(g^*)$  and  $g^* = \chi_C^\vee$ ) does.*

#### 4. Approximation orders and density orders

In this section we give a complete characterization of approximation orders and density orders from the sequence  $\{\mathcal{S}_h\}_h$  of shift-invariant spaces. In view of Theorem 3.12, we need only to consider the special case when each  $\mathcal{S}_h$  is principal. For  $\phi \in L_2(\mathbb{R}^d)$ , we let  $\Lambda_\phi \in L_\infty(C)$  be defined as in the introduction:

$$\Lambda_\phi := \left(1 - \frac{|\widehat{\phi}|^2}{[\widehat{\phi}, \widehat{\phi}]}\right)^{1/2}, \quad \text{on } C.$$

In terms of this  $\Lambda_\phi$ , (2.21) gives that

$$(4.1) \quad E(f, \mathcal{S}(\phi)) = (2\pi)^{-d/2} \|\widehat{f} \Lambda_\phi\| \quad \text{if } \text{supp } \widehat{f} \subset C.$$

For  $f \in L_2(\mathbb{R}^d)$  with  $\widehat{f}$  not just supported in  $C$ , we estimate  $E(f, \mathcal{S}(\phi)) = (2\pi)^{-d/2} E(\widehat{f}, \widehat{\mathcal{S}(\phi)})$  with the aid of Corollary 3.10 and the simple observation that

$$|E(\widehat{f}, \widehat{\mathcal{S}}) - E(\chi_C \widehat{f}, \widehat{\mathcal{S}})| \leq \|(1 - \chi_C) \widehat{f}\|$$

for an arbitrary subspace  $\mathcal{S}$  of  $L_2(\mathbb{R}^d)$ . Indeed, with the aid of (3.7), this estimate implies that

$$\begin{aligned} |E(f, \mathcal{S}^h) - (h/2\pi)^{d/2} E(\chi_C \widehat{f}_h, \widehat{\mathcal{S}})| &= |h^{d/2} E(f_h, \mathcal{S}) - (h/2\pi)^{d/2} E(\chi_C \widehat{f}_h, \widehat{\mathcal{S}})| \\ &= (h/2\pi)^{d/2} |E(\widehat{f}_h, \widehat{\mathcal{S}}) - E(\chi_C \widehat{f}_h, \widehat{\mathcal{S}})| \leq (h/2\pi)^{d/2} \|(1 - \chi_C) \widehat{f}_h\|. \end{aligned}$$

Therefore, Corollary 3.10 establishes

$$(4.2) \quad |E(f, \mathcal{S}^h) - (h/2\pi)^{d/2} E(\chi_C \widehat{f}_h, \widehat{\mathcal{S}})| \leq \varepsilon_f(h) h^k \|f\|_{W_2^k(\mathbb{R}^d)}.$$

**Theorem 4.3.** For  $\{\phi_h\}_h \subset L_2(\mathbb{R}^d)$ , the sequence  $\{\mathcal{S}(\phi_h)\}_h$  provides approximation order  $k$  if and only if

$$\left\{ \frac{\Lambda_{\phi_h}}{(h + |\cdot|)^k} \right\}_h$$

is bounded in  $L_\infty(C)$ .

**Remark.** Since each  $\Lambda_{\phi_h}$  is non-negative and bounded above by 1, and since each  $(h + |\cdot|)^k$  is bounded below by  $h^k$ , it is clear that each  $\frac{\Lambda_{\phi_h}}{(h + |\cdot|)^k}$  is an element of  $L_\infty(C)$ . So it is the *uniform* boundedness of  $\frac{\Lambda_{\phi_h}}{(h + |\cdot|)^k}$  as  $h \rightarrow 0$  that characterizes the approximation order  $k$ .

**Proof:** In view of (4.2),  $\{\mathcal{S}(\phi_h)\}_h$  provides approximation order  $k$  if and only if there exists some constant  $c$  such that for every  $f \in W_2^k(\mathbb{R}^d)$  and every  $h > 0$

$$(4.4) \quad h^{d/2} E(\chi_C \widehat{f}_h, \widehat{\mathcal{S}(\phi_h)}) \leq ch^k \|f\|_{W_2^k(\mathbb{R}^d)}.$$

Since  $\chi_C \widehat{f}_h$  is supported in  $C$ , we may appeal to (4.1) (i.e., to Theorem 2.20) to conclude that

$$(4.5) \quad \begin{aligned} h^d E(\chi_C \widehat{f}_h, \widehat{\mathcal{S}(\phi_h)})^2 &= h^d \int_C |\widehat{f}_h|^2 \Lambda_{\phi_h}^2 \\ &= h^{-d} \int_C |\widehat{f}(\cdot/h)|^2 \Lambda_{\phi_h}^2 = \int_{C/h} |\widehat{f}|^2 \Lambda_{\phi_h}(h \cdot)^2. \end{aligned}$$

For  $f \in W_2^k(\mathbb{R}^d)$ , the function  $\nu := (1 + |\cdot|)^k \widehat{f}$  is in  $L_2(\mathbb{R}^d)$ , and  $\|f\|_{W_2^k(\mathbb{R}^d)} = (2\pi)^{-d/2} \|\nu\|$ . With the aid of  $\nu$ , the last expression in (4.5) can be rewritten as

$$\int_{C/h} |\nu|^2 \frac{\Lambda_{\phi_h}(h \cdot)^2}{(1 + |\cdot|)^{2k}}.$$

Further, when  $f$  varies over all of  $W_2^k(\mathbb{R}^d)$ ,  $\nu$  varies over all of  $L_2(\mathbb{R}^d)$ , i.e.,  $g := |\nu|^2$  varies over all non-negative functions in  $L_1(\mathbb{R}^d)$ . This means that the  $k$ -approximation order requirement is equivalent to the existence of  $c > 0$  such that

$$(4.6) \quad \int_{\mathbb{R}^d} |g| \chi_{C/h} \frac{\Lambda_{\phi_h}(h \cdot)^2}{(1 + |\cdot|)^{2k}} \leq ch^{2k} \|g\|_{L_1(\mathbb{R}^d)}, \quad \forall h > 0, \quad \forall g \in L_1(\mathbb{R}^d).$$

Fixing  $h$ , the last condition states that  $\chi_{C/h} \frac{\Lambda_{\phi_h}(h \cdot)^2}{(1 + |\cdot|)^{2k}}$ , considered as a linear functional on  $L_1(\mathbb{R}^d)$ , is bounded by  $ch^{2k}$ . Consequently, having  $\{\mathcal{S}(\phi_h)\}_h$  provide approximation order  $k$  is equivalent to the existence of  $c > 0$  such that

$$\left\| \frac{\Lambda_{\phi_h}(h \cdot)}{(1 + |\cdot|)^k} \right\|_{L_\infty(C/h)} \leq ch^k.$$

The proof is thus completed, since upon rescaling the last condition becomes

$$(4.7) \quad \left\| \frac{\Lambda_{\phi_h}}{(h + |\cdot|)^k} \right\|_{L_\infty(C)} \leq c.$$

□

**Proof of Theorem 1.6.** In the case of this theorem,  $\phi_h = \phi$  for all  $h > 0$ . Using this in (4.7) and letting  $h \rightarrow 0$ , we get that (4.7) is equivalent to  $|\cdot|^{-k} \Lambda_\phi \in L_\infty(C)$ . □

**Remark.** Note that the cube  $C$  that appears in the characterization of approximation orders is entirely incidental. Since, for every  $h$ ,  $\Lambda_{\phi_h}$  is bounded by 1, and also  $(h + |\cdot|)^{-k}$  is bounded, independently of  $h$ , in any complement of a neighborhood of the origin, the cube  $C$  can be replaced by *any* neighborhood of the origin.

Another remark concerns the case  $k = 0$  which will soon be considered in the context of density orders. We have not discussed approximation order 0 simply because of lack of any mathematical interest: the requirement in this case is vacuous. This is in agreement with Theorem 4.3, for the boundedness of  $\{\frac{\Lambda_{\phi_h}}{(1+|\cdot|)^a}\}_h$  is also a vacuous condition, since each  $\Lambda_{\phi_h}$  is uniformly bounded by 1. This means that the statement of Theorem 4.3 is valid also for  $k = 0$ .

With Theorem 4.3 in hand, we turn our attention to the characterization of density orders. Our result concerning density orders is as follows.

**Theorem 4.8.** *For  $\{\phi_h\}_h \subset L_2(\mathbb{R}^d)$ , the sequence  $\{\mathcal{S}(\phi_h)\}_h$  provides density order  $k$  if and only if*

$$\left\{ \frac{\Lambda_{\phi_h}}{(h + |\cdot|)^k} \right\}_h$$

is bounded in  $L_\infty(C)$ , and

$$(4.9) \quad \lim_{h \rightarrow 0} h^{-d} \int_{h_a C} \frac{\Lambda_{\phi_h}^2}{(h + |\cdot|)^{2k}} = 0, \quad \forall a > 0.$$

**Proof:** In view of Theorem 4.3 and the definition of density orders, the theorem here is proved as soon as we show that, under the assumption that  $\{\frac{\Lambda_{\phi_h}}{(h+|\cdot|)^k}\}_h$  is bounded, the condition

$$(4.10) \quad \lim_{h \rightarrow 0} h^{d/2-k} E(f_h, \mathcal{S}(\phi_h)) = 0, \quad \forall f \in W_2^k(\mathbb{R}^d)$$

is equivalent to (4.9). For this we can follow the proof of Theorem 4.3 up to (4.6) to conclude that (4.10) is equivalent to the condition that

$$(4.11) \quad \lim_{h \rightarrow 0} h^{-2k} \int_{\mathbb{R}^d} |g| \chi_{C/h} \frac{\Lambda_{\phi_h}(h \cdot)^2}{(1 + |\cdot|)^{2k}} = 0, \quad \forall g \in L_1(\mathbb{R}^d).$$

Choosing  $g := \chi_{aC}$  in (4.11) and rescaling, we obtain (4.9), so that the *necessity* of (4.9) for  $k$ -density order is proved.

To prove the sufficiency, we define

$$\lambda_h := h^{-2k} \chi_{C/h} \frac{\Lambda_{\phi_h}(h \cdot)^2}{(1 + |\cdot|)^{2k}}, \quad h > 0.$$

We view the  $\lambda_h$  as elements of  $L_1(\mathbb{R}^d)^*$ . We want to show that (4.11) holds, namely that  $\{\lambda_h\}_h$  converges weak-\* to 0. We know that  $\{\lambda_h\}_h$  are positive, uniformly bounded, and by (4.9),  $\lambda_h(\chi_{aC}) \rightarrow 0$  for every  $a > 0$ . This latter condition implies that  $\lambda_h(\chi_K) \rightarrow 0$  for any compact  $K$ . By linearity,  $\lambda_h(g)$  tends to 0 for each compactly supported simple function  $g$ . Since such functions are dense in  $L_1(\mathbb{R}^d)$ , we obtain (4.11).  $\square$

**Proof of Theorem 1.7.** Since  $(h + |\cdot|)^{-2k} \leq |\cdot|^{-2k}$ , (1.8) implies that

$$\lim_{h \rightarrow 0} h^{-d} \int_{hC} \frac{\Lambda_{\phi_h}^2}{(h + |\cdot|)^{2k}} = 0,$$

which is the case  $a = 1$  in (4.9), and implies the rest of (4.9), since here  $\phi_h = \phi$  for all  $h$ , hence  $\Lambda_\phi$  does not change with  $h$ . Thus, Theorem 4.8 implies the sufficiency of (1.8).

On the other hand, if  $\mathcal{S}(\phi)$  provides density order  $k$ , then (4.9) holds (with  $\Lambda_{\phi_h} = \Lambda_\phi$ , all  $h$ ). Since  $|y|^{-2k} \leq c(h + |y|)^{-2k}$  for  $y \in hC \setminus (hC/2)$  and some absolute constant  $c$ , we obtain from (4.9) (with  $a = 1$ )

$$(4.12) \quad \int_{hC \setminus (hC/2)} \frac{\Lambda_\phi^2(y)}{|y|^{2k}} \leq \varepsilon(h) h^d$$

where  $\lim_{h \rightarrow 0} \varepsilon(h) = 0$ . Summing these estimates gives

$$(4.13) \quad \int_{hC} \frac{\Lambda_\phi^2(y)}{|y|^{2k}} \leq \sum_{j \geq 0} \varepsilon(2^{-j}h) 2^{-jd} h^d \leq 2 \max_{0 < u \leq h} \varepsilon(u) h^d.$$

Since the right side of (4.13) is  $o(h^d)$ , we obtain the necessity of (1.8).  $\square$

Combining the two last theorems with Theorem 3.12, we obtain

**Theorem 4.14.** *Let  $\{\mathcal{S}_h\}$  be a sequence of shift-invariant spaces. For each  $h$ , let  $\phi_h$  be the best approximation from  $\mathcal{S}_h$  to  $g^* = \chi_C^\vee$ . Then,  $\{\mathcal{S}_h\}_h$  provides approximation order  $k$  if and only if*

$$\left\{ \frac{\Lambda_{\phi_h}}{(h + \|\cdot\|)^k} \right\}_h$$

*is bounded in  $L_\infty(C)$ , and  $\{\mathcal{S}_h\}_h$  are  $k$ th-order dense if and only if, in addition to the above,*

$$(4.15) \quad \lim_{h \rightarrow 0} h^{-d} \int_{haC} \frac{\Lambda_{\phi_h}^2}{(h + \|\cdot\|)^{2k}} = 0, \quad \forall a > 0.$$

**Proof of Theorem 1.9.** This follows from Theorem 1.6, Theorem 1.7, and the reduction to the principal shift-invariant case given by Theorem 3.12 (with  $\phi_h^* = \phi^* = P_{\mathcal{S}} g^*$  for all  $h$ ).  $\square$

## 5. The Strang-Fix conditions

As mentioned in the introduction, approximation orders from the scaled spaces  $\{\mathcal{S}^h\}_h$  were characterized in [SF] under the assumptions that (a) the space  $\mathcal{S}^h$  is obtained as the  $h$ -dilate of the *same principal* shift-invariant space  $\mathcal{S}(\phi)$ ; (b) the generator  $\phi$  of  $\mathcal{S}(\phi)$  is compactly supported; and (c) the approximation order is realized in a controlled manner. The controlled approximation assumption, in turn, forces the condition  $\widehat{\phi}(0) \neq 0$ .

In order to compare these conditions to the characterization of approximation orders for principal shift-invariant spaces that we obtain in the present paper, we assume in this section that we have in hand a sequence  $\{\mathcal{S}(\phi_h)\}_h$  of principal shift-invariant spaces which satisfy one or both of the following conditions, in which  $\Omega$  is some neighborhood of the origin, and  $\eta$  and  $\mu$  are positive constants:

$$(5.1) \quad \exists \Omega, \mu, h_0 \quad \text{s.t.} \quad |\widehat{\phi}_h(x)| \leq \mu \quad \text{a.e. on } \Omega, \forall 0 < h < h_0;$$

$$(5.2) \quad \exists \Omega, \eta, h_0 \quad \text{s.t.} \quad \eta \leq |\widehat{\phi}_h(x)| \quad \text{a.e. on } \Omega, \forall 0 < h < h_0.$$

Note that, in case  $\phi_h$  does not change with  $h$  (i.e., when assumption (a) above holds), and  $\widehat{\phi}$  is continuous at the origin (e.g.,  $\phi$  is compactly supported, as in assumption (b) above), (5.1) is satisfied automatically and (5.2) is reduced to the mere condition

$$(5.3) \quad \widehat{\phi}(0) \neq 0.$$

We recall (see the remark after the proof of Theorem 1.6) that the uniform boundedness required in Theorem 4.3 for  $k$ -approximation order can be checked in any neighborhood  $\Omega$  of the origin, hence we can replace the cube  $C$  in the theorem by  $\Omega$ . As the next results show,  $\Lambda_{\phi_h}$  can often be replaced by

$$(5.4) \quad M_h := \left( \sum_{\beta \in 2\pi\mathbb{Z}^d \setminus 0} |\widehat{\phi}_h(\cdot + \beta)|^2 \right)^{1/2} = ([\widehat{\phi}_h, \widehat{\phi}_h] - |\widehat{\phi}_h|^2)^{1/2}.$$

**Lemma 5.5.** *If (5.1) holds and the sequence  $\{\mathcal{S}(\phi_h)\}_h$  provides approximation order  $k$ , then*

$$(5.6) \quad \left\{ \frac{M_h}{(h + |\cdot|)^k} \right\}_{h < h'_0}$$

is bounded in  $L_\infty(\Omega')$  for some 0-neighborhood  $\Omega'$  and some  $h'_0 > 0$ . On the other hand, if (5.2) holds and (5.6) is bounded in  $L_\infty(\Omega')$  for some 0-neighborhood  $\Omega$  and some  $h'_0$ , then  $\{\mathcal{S}(\phi_h)\}_h$  provides approximation order  $k$ .

**Proof:** If  $\{\mathcal{S}(\phi_h)\}_h$  provides approximation order  $k$ , then, by Theorem 4.3,  $\{(h + |\cdot|)^{-k} \Lambda_{\phi_h}\}_h$  is bounded, say by  $c$ , on  $\Omega$ . This, together with (5.1), implies that

$$(5.7) \quad (h + |\cdot|)^{-2k} M_h^2 \leq c(M_h^2 + |\widehat{\phi}_h|^2) \leq c(M_h^2 + \mu^2),$$

and therefore,

$$((h + |\cdot|)^{-2k} - c) M_h^2 \leq c\mu^2.$$

Thus, for sufficiently small  $h$  and some neighborhood  $\Omega' \subset \Omega$  of the origin, the leftmost term in (5.7) does not exceed  $2c\mu^2$ .

Conversely, (5.2) implies that, on  $\Omega$ ,

$$\Lambda_{\phi_h}^2 = 1 - \frac{|\widehat{\phi}_h|^2}{M_h^2 + |\widehat{\phi}_h|^2} \leq \frac{M_h^2}{|\widehat{\phi}_h|^2} \leq \eta^{-2} M_h.$$

Therefore, by Theorem 4.3, the boundedness of (5.6) implies that  $\{\mathcal{S}(\phi_h)\}_h$  provides approximation order  $k$ .  $\square$

We now consider in more detail necessary conditions for approximation order which follow from our characterization of approximation order. Since  $|\widehat{\phi}_h(\cdot + \beta)| \leq M_h$  for all  $\beta \in 2\pi\mathbb{Z}^d \setminus 0$ , the next theorem is a direct consequence of the last lemma:

**Theorem 5.8.** *If (5.1) holds and  $\{\mathcal{S}(\phi_h)\}_h$  provides approximation order  $k$ , then, for all  $0 < h < h_0$  and for all  $\beta \in 2\pi\mathbb{Z}^d \setminus 0$ , and in some 0-neighborhood,*

$$|\widehat{\phi}_h(\cdot + \beta)| \leq c(h + |\cdot|)^k,$$

for some  $c$  independent of  $\beta$  and  $h$ .

In case  $\widehat{\phi}$  does not change with  $h$ , we may let  $h \rightarrow 0$  in the last display and so obtain Theorem 1.14. This shows that the *necessity* of the Strang-Fix conditions (1.11) for  $k$ -approximation order holds in a very general setting. This is remarkable, since this implication is considered to be the “harder” one. An analogous  $L_\infty$ -result has been obtained in [BR2] by other means.

We now consider in more detail sufficient conditions for approximation order. There is no reason to believe that (upon assuming (5.2)) the assumptions

$$(5.9) \quad D^\gamma \widehat{\phi} = 0 \quad \text{on } 2\pi\mathbb{Z}^d \setminus 0 \quad \text{for all } |\gamma| < k$$

would suffice for approximation order  $k$  since from Lemma 5.5 we only can deduce the following:

**Corollary 5.10.** *If  $0 < \eta \leq |\widehat{\phi}|$  a.e. on some neighborhood  $\Omega$  of the origin, and if*

$$(5.11) \quad \sum_{\beta \in 2\pi\mathbb{Z}^d \setminus 0} |\widehat{\phi}(\cdot + \beta)|^2 \leq c|\cdot|^{2k}, \quad \text{a.e. on } \Omega,$$

then  $\mathcal{S}(\phi)$  provides approximation order  $k$ .

However, assumptions like (5.9) can only imply that, for each individual  $\beta \in 2\pi\mathbb{Z}^d \setminus 0$ ,

$$|\widehat{\phi}(\cdot + \beta)|^2 \leq c_\beta |\cdot|^{2k},$$

hence will not in general yield (5.11). On the other hand, there are several results in the literature which show that, under additional assumptions on  $\phi$ , (5.9) does imply that  $\mathcal{S}(\phi)$  provides approximation order  $k$ . For example, standard polynomial reproduction/quasi-interpolation arguments show that if

$$(5.12) \quad |\phi(x)| = O(|x|^{-k-d-\varepsilon}), \quad \text{as } x \rightarrow \infty,$$

and if  $\widehat{\phi}(0) \neq 0$ , then (5.9) implies that  $\mathcal{S}(\phi)$  provides approximation order  $k$  (cf. e.g., Proposition 1.1 and Corollary 1.2 in [DJLR]). Unfortunately, the decay conditions (5.12) fail to hold for many functions  $\phi$  of interest (primarily radial basis functions, and usually because  $\widehat{\phi}$  is not smooth enough at 0), and in such a case, the polynomial reproduction argument either fails, or is not easily converted into approximation orders. Circumventing the polynomial reproduction argument was actually the major objective of [BR2]. In our context, Theorem 1.6 leads to a remarkable result, which allows (5.12) to be replaced by a much weaker condition, and which we now describe.

For this result, we need a local version  $W_2^\rho(\Omega)$  of the potential spaces  $W_2^\rho(\mathbb{R}^d)$ . If  $\rho$  is an integer, then this space is simply the Sobolev space of all functions whose (weak) derivatives up to order  $\rho$  (inclusive) are in  $L_2(\Omega)$ . In this case, if  $\{\Omega_\beta\}_{\beta \in I}$  is a disjoint collection of open subsets of  $\mathbb{R}^d$ , we have  $\sum_{\beta \in I} \|f\|_{W_2^\rho(\Omega_\beta)}^2 = \|f\|_{W_2^\rho(\cup_{\beta \in I} \Omega_\beta)}^2$ . As is well-known, there are several equivalent extensions of the definition of  $W_2^\rho(\Omega)$  to the case of a fractional  $\rho$  (see, e.g., [A; ch.7]). For fractional  $\rho$ , we have the following subadditivity property:

$$(5.13) \quad \sum_{\beta \in I} \|f\|_{W_2^\rho(\Omega_\beta)}^2 \leq c \|f\|_{W_2^\rho(\cup_{\beta \in I} \Omega_\beta)}^2,$$

whenever, say,  $\{\Omega_\beta\}_\beta$  is a disjoint collection of cubes; (cf. [A; p.225]). Our result is as follows:

**Theorem 5.14.** Assume that  $0 < \eta \leq \widehat{\phi}$  a.e. on some cube  $\Omega$  centered at the origin. Let  $A := \cup_{\beta \in 2\pi\mathbb{Z}^d \setminus 0} (\Omega + \beta)$ . If  $\widehat{\phi} \in W_2^\rho(A)$  for some  $\rho > k + d/2$ , and if (5.9) holds, then  $\mathcal{S}(\phi)$  provides approximation order  $k$ .

The virtue of this theorem is that we can take  $\Omega$  to be so small that  $A$  does not contain the origin. This is important since in many cases of interest  $\widehat{\phi}$  is smooth on  $\mathbb{R}^d \setminus 0$  but has some singularity at the origin (this happens, e.g., when  $\phi$  is obtained by the application of a difference operator to a fundamental solution of an elliptic equation). But, if  $\phi$  satisfies (5.12), then  $\widehat{\phi}$  is *globally* smooth, since we obtain from (5.12) that  $\widehat{\phi} \in W_2^\rho(\mathbb{R}^d)$  for  $\rho = k + d/2 + \varepsilon/2$  as well as  $\widehat{\phi} \in C^k(\mathbb{R}^d)$ . Thus, Theorem 5.14 and Theorem 1.14 together imply the following result.

**Corollary 5.15.** If  $\phi$  satisfies (5.12) and  $\widehat{\phi}(0) \neq 0$ , then  $\mathcal{S}(\phi)$  provides approximation order  $k$  if and only if (5.9) holds.

**Proof of Theorem 5.14.** It follows from (5.9) that, for every  $\beta \in 2\pi\mathbb{Z}^d \setminus 0$ , and with  $\Omega_\beta := \Omega + \beta$ ,

$$(5.16) \quad |\widehat{\phi}(x + \beta)| \leq c|x|^k \max_{|\gamma|=k} \|D^\gamma \widehat{\phi}\|_{L^\infty(\Omega_\beta)}, \quad \text{for } x \in \Omega.$$

Since  $\rho > k + d/2$ , the Sobolev embedding theorem (cf. [A; p.217]) implies that  $W_2^\rho(\Omega_\beta)$  is continuously embedded in the Sobolev space  $W_\infty^k(\Omega_\beta)$ . Thus,

$$\max_{0 \leq |\gamma| \leq k} \|D^\gamma \widehat{\phi}\|_{L^\infty(\Omega_\beta)} \leq c_1 \|\widehat{\phi}\|_{W_2^\rho(\Omega_\beta)},$$

with  $c_1$  independent of  $\beta$  (since all the  $\Omega_\beta$  are translates of each other). Substituting this into (5.16) we obtain that

$$|\widehat{\phi}(x + \beta)| \leq c_2 |x|^k \|\widehat{\phi}\|_{W_2^\rho(\Omega_\beta)}, \quad x \in \Omega, \beta \in 2\pi\mathbb{Z}^d \setminus 0.$$

Squaring the last inequality and summing over  $\beta \in 2\pi\mathbb{Z}^d \setminus 0$ , we obtain, in view of (5.13), that

$$\sum_{\beta \in 2\pi\mathbb{Z}^d \setminus 0} |\widehat{\phi}(x + \beta)|^2 \leq c_3 |x|^{2k} \|\widehat{\phi}\|_{W_2^\rho(A)}^2.$$

Lemma 5.5 now supplies the conclusion that  $\mathcal{S}(\phi)$  provides approximation order  $k$ . □

In applications, it might be convenient to take  $\rho$  to be the least integer that satisfies  $\rho > k + d/2$ . For this case, Theorem 5.14 reduces to Theorem 1.15.

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