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MULTIVARIATE TRIGONOMETRIC POLYNOMIAL APPROXIMATIONS WITH FREQUENCIES FROM THE HYPERBOLIC CROSS

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1. Introduction. While univariate approximation by trigonometric polynomials is one of the most classical and fully developed chapters in approximation theory, the situation regarding multivariate trigonometric polynomial approximation on the d-dimensional torus $\mathbb{T}^d$, $d \geq 2$, is less settled. To a considerable extent this is connected to the fact that there are various possibilities for restriction of the frequencies of these polynomials. If S is a subset of $\mathbb{Z}^d$, and $1 \leq p \leq \infty$, then we denote by

$$\mathcal{T}(S,p) := \overline{\operatorname{span}}\{e_k : k \in S\}$$

the $L_p(\mathbb{T}^d)$-closure of the span of the exponentials e_k, $k \in S$, where

$$e_k(x) := e^{ik\cdot x} = e^{i(k_1x_1+\cdots+k_dx_d)}, \quad x \in \mathbb{R}^d.$$

In the case where S is a finite set, these spaces do not depend on p and are merely the algebraic span of corresponding exponentials. We denote this space by $\mathcal{T}(S)$.

The case when the frequencies come from the d-cube $S := S_n := \{k : -n \leq k_j \leq n,\ j = 1, \ldots, d\}$ can be treated very simply from the univariate case (see, e.g., [1]). However, other restrictions on the frequencies arise in a natural manner and frequently have important applications. In the present paper we consider one of these cases where the frequencies are restricted to come from the hyperbolic cross region

$$\Gamma(n) := \{k : \prod_{j=1}^{d} \max(1, |k_j|) \leq n\}$$

or from the related region

$$\Gamma'(n) := \{k : \prod_{j=1}^{d} |k_j| \leq n\}.$$

These two regions differ only in the frequencies k which have some zero components. Namely, $\Gamma'(n)$ contains all k such that $\prod_{j=1}^{n} k_j = 0$. While the set $\Gamma(n)$ is of most interest to us in the present paper, the set $\Gamma'(n)$ is useful as a vehicle in establishing a theorem on the approximation by polynomials from $\mathcal{T}(\Gamma(n))$. If p is fixed, then we use the abbreviated notation

$$\mathcal{T}_n := \mathcal{T}(\Gamma(n)), \quad \mathcal{T}_n' := \mathcal{T}(\Gamma'(n), p).$$

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For any nonnegative integer n, we denote by

$$E_n(f)_p := \inf_{T\in\mathcal{T}_n} \|f-T\|_p, \quad E'_n(f)_p := \inf_{T\in\mathcal{T}'_n} \|f-T\|_p$$

the error in L_p-approximation by elements of $\mathcal{T}_n$ (respectively, $\mathcal{T}'_n$). Here and throughout the paper

$$\|f\|_p := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} |f(x)|\,dx$$

denotes the $L_p(\mathbb{T}^d)$ norm.

Hyperbolic cross approximation was initiated by Babenko [2] and the first important results were also obtained by Telyakovskii [3] and Mityagin [4]. The recent monographs of Temlyakov [5, 6] give the detailed description of the history of the problem. One of the most important properties of hyperbolic cross approximation is that the finite dimensional space $\mathcal{T}_n$ is optimal in the simultaneous approximation of smoothness classes of functions (see Sec. 5 of Chapter 3 in [6]).

We are interested in the smoothness properties of f which govern its approximation order $E_n(f)_p$. We can classify functions according to the speed that $E_n(f)_p$ converges to 0. For example, given $\alpha > 0$ and $0 < q \le \infty$, we denote by $\mathcal{A}^\alpha_q(L_p(\mathbb{T}^d)) := \mathcal{A}^\alpha_q(L_p(\mathbb{T}^d), (\mathcal{T}_n))$ the set of functions $f \in L_p(\mathbb{T}^d)$ such that

$$|f|_{A^\alpha_q(L_p(\mathbb{T}^d))} := \begin{cases} \left(\sum_{k=0}^{\infty}[2^{k\alpha}E_{2^k}(f)_p]^q\right)^{1/q} & 0<q<\infty, \\ \sup_{0\le k<\infty} 2^{k\alpha}E_{2^k}(f)_p & q=\infty, \end{cases} \tag{1.1}$$

is finite. We define a "norm" on $A^\alpha_q(L_p(\mathbb{T}^d))$ by

$$\|f\|_{A^\alpha_q(L_p(\mathbb{T}^d))} := \|f\|_p + |f|_{A^\alpha_q(L_p(\mathbb{T}^d))}. \tag{1.2}$$

By replacing E_{2^k} by E'_{2^k}, we obtain the corresponding class $\mathcal{A}^\alpha_q(L_p(\mathbb{T}^d), (\mathcal{T}'_n))$ for $(\mathcal{T}'_n)$. The main purpose of the present paper is to provide a characterization of the spaces $\mathcal{A}^\alpha_q(L_p(\mathbb{T}^d))$ by introducing new moduli of smoothness for functions in $L_p(\mathbb{T}^d)$.

Previous research on hyperbolic cross approximation has shown the importance of measuring the smoothness of functions by means of mixed derivatives and mixed differences. For $r = 1, 2, \ldots$, we denote by $\mathbb{D}^r := \frac{\partial^r}{\partial x_1^r}\cdots\frac{\partial^r}{\partial x_d^r}$ the mixed partial derivative of order r. In this case, for $1 < p < \infty$, Mityagin probed [4] the following estimates:

$$E'_n(f)_p \le C(r,p)\|\mathbb{D}^r f\|_p n^{-r}, \quad n = 1, 2, \ldots\ . \tag{1.3}$$

There is a corresponding Bernstein type inequality for $\mathcal{T}'_n$ (the analogue of which for $\mathcal{T}_n$ was given by Mityagin [4]):

$$\|\mathbb{D}^r T\|_p \le C(r,p)\|T\|_p n^r, \quad T \in \mathcal{T}'_n,\ n = 1, 2, \ldots\ . \tag{1.4}$$

It is well known (see, e.g., [7]) that the Jackson and Bernstein type inequalities (1.3), (1.4) can be used to characterize the function classes $\mathcal{A}^\alpha_q(L_p(\mathbb{T}^d), (\mathcal{T}'_n))$. Namely, let $W^r(L_p(\mathbb{T}^d))$ denote the Sobolev space consisting of all functions $f \in L_p(\mathbb{T}^d)$ for which the distributional derivative $\mathbb{D}^r f$ is in $L_p(\mathbb{T}^d)$, and let $K(f,t)$ denote the K-functional for the pair $L_p(\mathbb{T}^d)$, $W^r(L_p(\mathbb{T}^d))$:

$$K(f,t) := K(f,t,L_p(\mathbb{T}^d), W^r(L_p)) := \inf_{g\in W^r(L_p(\mathbb{T}^d))}\{\|f-g\|_p + t^r\|\mathbb{D}^r g\|_p\}, \quad t>0. \tag{1.5}$$

Then, for $0 < \alpha < r$ and $0 < q < \infty$, the space $\mathcal{A}^\alpha_q(L_p(\mathbb{T}^d), (\mathcal{T}'_n))$ consists of functions f such that

$$\left(\int_0^\infty [t^{-\alpha}K(f,t)]^q\,dt/t\right)^{1/q} < \infty. \tag{1.6}$$

A similar statement holds for $q = \infty$. The problem with this characterization is that we do not know the K-functional (1.5) and therefore the characterization (1.6) essentially replaces one unknown quantity $(E'_{2^k}(f)_p)$ by another $K(f,t)$.

Our main result, formulated in Theorem 4.4, characterizes $\tilde{\mathcal{A}}^\alpha_q(L_p(\mathbb{T}^d),(\mathcal{T}_n))$ by replacing $E_{2^k}(f)_p$ in (1.1) by $\Omega^*_r(f,2^{-k})_p$, where Ω^*_r is a certain modulus of smoothness. To prove this theorem, we first establish in Sec. 3 a similar characterization of $\mathcal{A}^\alpha_q(L_p(\mathbb{T}^d),(\mathcal{T}'_n))$. In the introduction we describe only this last characterization, for it is easier.

For a real number s, let $\tilde{\Delta}(s)$ be the symmetric univariate difference operator defined for a univariate function f by

$$\tilde{\Delta}(s)f := \frac{1}{2}(f(x+s/2) - f(x-s/2)),$$

and for a positive integer r, let $\tilde{\Delta}^r(s)$ be its r-fold composition defined inductively by $\tilde{\Delta}^r(s) := \tilde{\Delta}(s)\tilde{\Delta}^{r-1}(s)$. From this, for $s \in \mathbb{R}^d$ and for an arbitrary multivariate function $f(x)$, $x \in \mathbb{T}^d$, we can define the difference

$$\tilde{\Delta}^r(s)f := \tilde{\Delta}^r(s_1)\cdots\tilde{\Delta}^r(s_d)f, \tag{1.7}$$

where the operator $\tilde{\Delta}^r(s_j)$ is applied to the variable x_j.

To take certain averages of $\tilde{\Delta}^r(s)$, we use the symmetric multivariate B-spline $M_l(x_1,\ldots,x_d) := M_l(x_1)\cdots M_l(x_d)$ formed from the symmetric univariate B-splines $M_l(x) := M(x;-l/2,\ldots,l/2)$, $x \in \mathbb{R}$, of order l (degree $l-1$) with knots at the points $-l/2, -l/2+1,\ldots,l/2$. Recall that the univariate M_l is an $(l-1)$-fold convolution of $\chi_{[-1/2,1/2]}$ with itself. It follows that $\int_{\mathbb{R}^d} M_l(x_1,\ldots,x_d)\,dx_1 \cdots dx_d = 1$. For $s \in \mathbb{R}^d$, $s_j > 0$, $j = 1,\ldots,d$, the dilated B-spline

$$M_l(x,s) := s_1^{-1}\ldots s_d^{-1} M_l(s_1^{-1}x_1,\ldots,s_d^{-1}x_d)$$

also has mean value 1.

We denote by $M_l^o(.,s)$ the 2π-periodization of $M_l(\cdot,s)$. Recall that if $g \in L_1(\mathbb{R}^d)$, then its 2π-periodization $g^o(x) := \sum_{k\in Z^d} g(x+2\pi k)$ belongs to $L_1(\mathbb{T}^d)$. For $s, s_j > 0$, $j = 1,\ldots,d$, we define the convolution operator $A_l(s)$ by means of

$$A_l(s)f := \int_{u\in\mathbb{T}^d} f(\cdot - u)M_l^o(u,s)dt. \tag{1.8}$$

For $r = 1,2,\ldots,$ we denote

$$l := l(r) := \begin{cases} 2 & \text{for } r \text{ even}, \\ 3 & \text{for } r \text{ odd}. \end{cases} \tag{1.9}$$

From this point on, l will be this number if it is not otherwise stated. Then

$$w_r(s)f := A_l(s)\tilde{\Delta}^r(s)f \tag{1.10}$$

is a convolution operator which can be viewed as the averaging of the rth mixed differences $\tilde{\Delta}^r(s)$. From $w_r(s)$, we construct for each $t > 0$ the operator $W_r(t)$ which is defined by

$$W_r(t)(f) := \int_0^\infty \cdots \int_0^\infty w_r(s_1,s_2,\ldots,s_{d-1},s_1^{-1}s_2^{-1}\cdots s_{d-1}^{-1}t)(f)\frac{ds_{d-1}}{s_{d-1}}\cdots\frac{ds_1}{s_1}. \tag{1.11}$$

In other words, the $s = (s_1,\ldots,s_d)$ appearing in the definition (1.11) satisfy the relation $s_1\cdots s_d = t$. Thus, $W_r(t)$ is a convolution operator for each $t > 0$. Finally, for $r = 1,2,\ldots$, $t > 0$, $1 < p < \infty$, we define the moduli of smoothness

$$\Omega_r(f,t)_p := \|W_r(t)f\|_p. \tag{1.12}$$

Remarks. The expression $\Omega_r(f,t)_p$ is not a modulus of smoothness in the usual sense; for example, it is not monotone increasing. Monotonicity could be obtained by redefining $\Omega_r(f,h)$ as the *sup* of the norms in (1.12) over all $0 \le t \le h$. Our results given later would hold equally well with this modification. We do not introduce the *sup* in the definition of $\Omega_r(f,t)_p$ in order to emphasize the fact that our results and methods do not need it.

In the definition of $w_r(s)$ and, as a result, in the definition of $\Omega_r(f,t)_p$, we used the operator $A_l(s)$ with $l = 2,3$, (see (1.9), (1.10)). In the results we prove in this paper that this role can be played equally well by any other $l \ge 2$ provided that $r + l$ is even. The selection $l = 1$ also makes sense. However, in this case $\Omega_r(f,t)_p$ is not bounded by $C\|f\|_p$ with C independent from t and our inverse theorem would contain a "tail." As we discuss in Sec. 5 the integration in (1.11) can be replaced by integration over a bounded region. However, it seems that definition (1.11) is notationally the most convenient and the proofs seem the clearest with this choice.

We shall prove in this paper the following characterization theorem for $\mathcal{A}_q^\alpha(L_p(\mathbb{T}^d)),(\mathcal{T}_n'))$.

Theorem 1.1. *Let $r = 1,2,\ldots$, $1 < p < \infty$, $0 < q \le \infty$, and $0 < \alpha < r$. Then $f \in \mathcal{A}_q^\alpha(L_p(\mathbb{T}^d),(\mathcal{T}_n'))$ if and only if the quantity*

$$\begin{cases} \left(\sum_{k=0}^\infty [2^{k\alpha}\Omega_r(f,2^{-k})_p]^q\right)^{1/q} & 0 < q < \infty, \\ \sup_{0 \le k < \infty} 2^{k\alpha}\Omega_r(f,2^{-k})_p & q = \infty, \end{cases} \tag{1.13}$$

is finite. Moreover, the quantity (1.13) *is equivalent to $\|f\|_{\mathcal{A}_q^\alpha(L_p(\mathbb{T}^d),(\mathcal{T}_n'))}$ after adding $\|f\|_p$.*

A similar result holds for approximation by the elements of $\mathcal{T}_n$ (see Sec. 4). The proof of Theorem 1.1 (given in Sec. 3) rests on the Littlewood–Paley theory and certain multiplier theorems which we develop in the following section.

In this paper we consider the approximation by elements of $\mathcal{T}_n'$ as an intermediary toward our ultimate interest of the approximation by elements of $\mathcal{T}_n$. There is another related approach to obtaining the results of Sec. 4 that we should mention and, in fact, we employ later in this paper. Rather than modifying the frequencies of the set $\Gamma(n)$ by considering $\Gamma'(n)$, we could modify the functions we are going to approximate, considering only functions of the space $L_p^0(\mathbb{T}^d)$ which consists of functions of $L_p(\mathbb{T}^d)$ whose Fourier coefficients satisfy the relation $\hat{f}(k) = 0$ for $k = (k_1,\ldots,k_d)$ such that $k_1 \cdots k_d = 0$. Equivalently, $L_p^0(\mathbb{T}^d)$ can be defined as the set of functions $f \in L_p(\mathbb{T}^d)$ such that

$$\int_{-\pi}^{\pi} f(x)dx_j = 0, \quad a.e, \quad j = 1,\ldots,d.$$

The results proved in this paper are restricted to $L_p(\mathbb{T}^d)$ approximation for $1 < p < \infty$. The reason for this is that the proofs are essentially based on such tools as the Littlewood–Paley theorem and Marcinkiewicz's multiplier theorem. It seems decidedly more difficult to establish similar results for the cases $p = 1,\infty$.

2. Littlewood–Paley theory and multiplier theorems. If $f \in L_1(\mathbb{T}^d)$, then its Fourier series is

$$f \sim \sum_{k\in\mathbb{Z}^d} \hat{f}(k)e_k$$

with the Fourier coefficients given by

$$\hat{f}(k) := (2\pi)^{-d}\int_{\mathbb{T}^d} f(x)e_{-k}(x)\,dx, \quad k \in \mathbb{Z}^d.$$

The Littlewood–Paley theorem gives a way of estimating the $L_p(\mathbb{T}^d)$-norm of f from its Fourier coefficients.

We denote by $\mathbb{Z}_+^d$ the set of nonnegative multi-integers. For $\mu \in \mathbb{Z}_+^d$, we put

$$\rho(\mu) := \{k \in \mathbb{Z}^d : 2^{\mu_j - 1} \le |k_j| < 2^{\mu_j}\},$$

and

$$\delta_\mu(f) := \sum_{k\in\rho(\mu)} \hat{f}(k)e_k. \tag{2.1}$$

Then a function $f \in L_p(\mathbb{T}^d)$ has the representation

$$f = \sum_{\mu\in\mathbb{Z}^d_+} \delta_\mu(f),$$

and the Littlewood–Paley theorem (see the book of Nikolskii [1, Sec. 1.5.2]) says that

$$\|f\|_p \approx \|(\sum_{\mu\in\mathbb{Z}^d_+} |\delta_\mu(f)|^2)^{1/2}\|_p, \quad 1 < p < \infty, \tag{2.2}$$

with the constants of equivalence dependent only on p and d. Here and in what follows, the statement $A \approx B$ means that $A \le C_1 B$ and $B \le C_2 A$; C_1, C_2 are absolute constants if not stated otherwise.

One of the most important applications of the Littlewood–Paley theorem is the Marcinkiewicz multiplier theorem which gives sufficient conditions on a sequence $\lambda := (\lambda(k))_{k\in\mathbb{Z}^d}$ for the linear operator

$$\Lambda(f) := \sum_{k\in\mathbb{Z}^d} \lambda(k)\hat{f}(k)e_k \tag{2.3}$$

to be a bounded mapping from $L_p(\mathbb{T}^d)$ into itself. One can at least formally view Λ as a convolution operator with kernel $\sum_{k\in\mathbb{Z}^d} \lambda(k)e_k$.

To describe Marcinkiewicz's theorem, we introduce the following notation for shift and difference operators applied to sequences $\lambda \in \mathbb{Z}^d$. Let $\tau(j)$ denote the shift operator defined by

$$\tau(j)\lambda(k) := \lambda(k + \delta_j), \quad k \in \mathbb{Z}^d,$$

where $\delta_j := (\delta_j(\nu))_{\nu=1}^d$ is the coordinate vector whose jth coordinate is equal to 1, and other coordinates are zero. Further, we put

$$\Delta(j) := \tau(j) - I,$$

where I the identity operator. More generally, if $V \subset \{1, \ldots, d\}$, we put

$$\tau(V)\lambda(k) := [\prod_{v\in V} \tau(v)]\lambda(k), \tag{2.4}$$

and

$$\Delta(V)\lambda(k) := [\prod_{v\in V} \Delta(v)]\lambda(k). \tag{2.5}$$

The operators $\tau(v)$ and $\Delta(v)$ act only on the vth coordinate of λ. We shall often use the fact that the operators $\Delta(V)$ and $\tau(V)$, $V \subset \{1, \ldots, d\}$, commute.

For a given $V \subset \{1, \ldots, d\}$, we put $V' := \{1, \ldots, d\} \setminus V$; thus, $V \cup V' = \{1, \ldots, d\}$. If V' is such a set, and $c := (c(v))_{v\in V'}$ is a sequence of integers, then the set

$$H(V, c) := \{k : k|_{V'} = c\}$$

is a lower dimensional hyperplane. We define

$$H(V, c, \rho(\mu)) := H(V, c) \cap \{k : k \in \rho(\mu) \text{ and } \tau(V)k \in \rho(\mu)\}.$$

Theorem 2.1. (Marcinkiewicz multiplier theorem). *If for a fixed constant $M > 0$, the sequence $(\lambda(k))_{k\in\mathbb{Z}^d}$ satisfies*

$$\sum_{k\in H(V,c,\rho(\mu))} |\Delta(V)\lambda(k)| \le M \tag{$*$}$$

for each $V \subset \{1,\ldots,d\}$, c, and $\mu \in \mathbb{Z}^d_+$, then Λ is a bounded operator in $L_p(\mathbb{T}^d)$, $1 < p < \infty$, and

$$\|\Lambda(f)\|_p \le C(p)M\|f\|_p, \quad f \in L_p(\mathbb{T}^d). \tag{2.6}$$

For the case $V = \varnothing$ inequality $(*)$ is equivalent to the boundedness of the sequence $(\lambda(k))$. The proof of Theorem 2.1 in two dimensions is in [8].

Remarks. In the sequel we often use the following remarks to the Marcinkiewicz theorem. Let $\mathbb{M}$ be a subset of $\mathbb{Z}^d_+$. If λ is zero outside $\cup_{\mu\in\mathbb{M}}\rho(\mu)$, then we only need check $(*)$ for $\mu \in \mathbb{M}$. Secondly, if λ satisfies $(*)$ for $k \in \cup_{\mu\in\mathbb{M}}\rho(\mu)$, then, for each $1 < p < \infty$, λ is a bounded multiplier on the space $X_{\mathbb{M}}$ which is the closure in $L_p(\mathbb{T}^d)$ of the finite linear combinations of the exponential e_k, $k \in \cup_{\mu\in\mathbb{M}}\rho(\mu)$.

We now consider the mapping properties of the operator $W_r(t)$ defined by (1.11). It is a convolution operator, and we first find its multiplier coefficients $\lambda(k,t)$, $k \in \mathbb{Z}^d$, $t > 0$.

The univariate operator $\tilde{\Delta}(s)$ is convolution with a Dirac measure of mass 1/2 at the point $s/2$, and of mass $-1/2$ at the point $-s/2$. Its multiplier coefficients are $i\sin ks/2$ for $k \ne 0$, and 0 for $k = 0$. Therefore, the r-fold difference $\tilde{\Delta}^r(s)$ has multiplier coefficients $(i\sin ks/2)^r$, $k \in \mathbb{Z}$. From this we derive that the multivariate operator $\tilde{\Delta}^r(s)$, $s = (s_1,\ldots,s_d)$, has multiplier coefficients $\prod_{j=1}^d (i\sin k_j s_j/2)^r$ for $\prod_{j=1}^d k_j \ne 0$, and 0 otherwise.

Similarly, the averaging of $w_r(s)$, $s = (s_1,\ldots,s_d)$, in definition (1.10) is a convolution with M_l^ρ, and it has multiplier coefficients $\prod_{j=1}^d (\frac{\sin k_j s_j/2}{k_j s_j/2})^l$; moreover, for $k_j = 0$ the corresponding terms in the product are replaced by 1. Thus, we obtain

$$w_r(s)f = \sum_{k\in\mathbb{Z}^d} \eta(k,s)\hat{f}(k)e_k, \tag{2.7}$$

where

$$\eta(k,s) := \begin{cases} i^{rd}\prod_{j=1}^d \frac{(\sin k_j s_j/2)^{r+l}}{(k_j s_j/2)^l} & \prod_{j=1}^d k_j \ne 0, \\ 0 & \prod_{j=1}^d k_j = 0. \end{cases} \tag{2.8}$$

It follows from (1.11) that the multiplier coefficients $\lambda(k,t)$ of $W_r(t)$ are given by

$$\lambda(k,t) = \int_0^\infty \cdots \int_0^\infty \eta(k,(s_1,s_2,\ldots,s_{d-1},s_1^{-1}s_2^{-1}\cdots s_{d-1}^{-1}t))\frac{ds_{d-1}}{s_{d-1}}\cdots\frac{ds_1}{s_1}. \tag{2.9}$$

Now we consider the action of $W_r(t)$ on each of the sums $\delta_\mu(f)$ given in (2.1). To analyze the mapping properties of the operators $W_r(t)\delta_\mu$, we estimate the differences which appear in $(*)$ of the Marcinkiewicz multiplier theorem. Let $\phi(x) := \frac{(\sin x/2)^{r+l}}{(x/2)^l}$. Then ϕ belongs to $C^\infty(\mathbb{R})$ and satisfies the relations

$$|\phi(x)| \le C\min(x^r, x^{-l}) =: Cm_0(x), \quad x \ge 0, \tag{2.10}$$

and

$$|\phi'(x)| \le C\min(x^{r-1}, x^{-l}) =: Cm_1(x), \quad x \ge 0, \tag{2.11}$$

where C is a constant dependent only on r, and l is defined by (1.9). Therefore, for any set $V \subset \{1,\ldots,d\}$, we have

$$|\Delta(V)\eta(k,s)| \le C\prod_{v\in V} s_v m_1((|k_v|+1)s_v)\prod_{v\in V'} m_0(|k_v|s_v). \tag{2.12}$$

The integral (1.11), defining $W_r(t)$, is symmetric with respect to the variables $s_1, \dots, s_d$. Therefore, in estimating the sums in the relation (*) of the Marcinkiewicz multiplier theorem, we can assume that $V = \{\ell+1, \dots, d\}$ and $V' = \{1, \dots, \ell\}$. We assume this form for V and V' from now on. Using (2.12) in (2.9), we obtain

$$|\Delta(V)\lambda(k,i)| \le C \int_0^\infty \cdots \int_0^\infty m_*((|k_d|+1)s_1^{-1}\cdots s_{d-1}^{-1}t) \times \prod_{j=1}^{\ell} m_0(|k_j|s_j) \prod_{j=\ell+1}^{d-1} s_j m_1((|k_j|+1)s_j)\frac{ds_{d-1}}{s_{d-1}}\cdots\frac{ds_1}{s_1}, \tag{2.13}$$

where $m_* := 1$ in the case $\ell = d$ (i.e., $V = \varnothing$), and $m_*(x) := (|k_d|+1)^{-1}xm_1(x)$ otherwise. When $\ell = d$, the term $m_0(|k_d|s_d)$ appearing in the first product is interpreted as $m_0(|k_d|s_1^{-1}\cdots s_{d-1}^{-1}t)$.

We now estimate the right side of (2.13). The following lemma bounds some of the integrals which arise in estimating (2.13). We use the abbreviated notation $L(x) := |\ln x| + 1$.

Lemma 2.2. *For $r = 1, 2, \dots$, $\alpha, \beta > 0$, and $\gamma = 0, 1, \dots$, we have*

$$\int_0^\infty m_0(\alpha x)m_0(\beta/x)[L(\beta/x)]^\gamma \frac{dx}{x} \le Cm_0(\alpha\beta)[L(\alpha\beta)]^{\gamma+1}, \tag{2.14}$$

$$\int_0^\infty x^{-1}m_0(\alpha x)m_1(\beta/x)[L(\beta/x)]^\gamma \frac{dx}{x} \le C\alpha m_1(\alpha\beta)[L(\alpha\beta)]^{\gamma+1}, \tag{2.15}$$

and

$$\int_0^\infty m_1(\alpha x)m_1(\beta/x)[L(\beta/x)]^\gamma \frac{dx}{x} \le Cm_1(\alpha\beta)[L(\alpha\beta)]^{\gamma+1}, \tag{2.16}$$

where C is a constant dependent only on r and γ.

Proof. We first prove (2.14). Assume that $\alpha\beta \le 1$. We expand the integral in the left side of (2.14) in the sum of three integrals over the intervals $[0,\beta]$, $[\beta, 1/\alpha]$, and $[1/\alpha, \infty)$. By definition of m_0, the first of these integrals is estimated as

$$\begin{aligned}\int_0^\beta (\alpha x)^r(\beta/x)^{-l}[L(\beta/x)]^\gamma \frac{dx}{x} &= \alpha^r\beta^{-l}\int_0^\beta x^{r+l}[L(\beta/x)]^\gamma\frac{dx}{x}\\ &= \alpha^r\beta^{-l}\int_0^1 (\beta u)^{r+l}[L(1/u)]^\gamma\frac{du}{u} = (\alpha\beta)^r\int_0^1 u^{r+l}[L(1/u)]^\gamma\frac{du}{u} \le C(\alpha\beta)^r,\end{aligned} \tag{2.17}$$

where we use the substitution $x = \beta u$ and the fact that the last integral is convergent.

Similarly, the integral over $[1/\alpha, \infty)$ is estimated as

$$\int_{1/\alpha}^\infty (\alpha x)^{-l}(\beta/x)^r[L(\beta/x)]^\gamma\frac{dx}{x} = \alpha^{-l}\beta^r\int_{1/\alpha}^\infty x^{-r-l}[L(\beta/x)]^\gamma\frac{dx}{x} \le C(\alpha\beta)^r[L(\alpha\beta)]^\gamma, \tag{2.18}$$

where the last inequality is easily proved by discretizing the integral. The second integral is estimated as

$$\int_\beta^{1/\alpha}(\alpha x)^r(\beta/x)^r[L(\beta/x)]^\gamma\frac{dx}{x} = (\alpha\beta)^r\int_\beta^{1/\alpha}[L(\beta/x)]^\gamma\frac{dx}{x} \le C(\alpha\beta)^r[L(\alpha\beta)]^{\gamma+1}. \tag{2.19}$$

Estimates (2.17)–(2.19) prove (2.14) in the case $\alpha\beta \le 1$, since $m_0(\alpha\beta) = (\alpha\beta)^r$ in this case.

In the case $\alpha\beta > 1$, we expand the integral in the left side of (2.14) in three integrals corresponding to the intervals $[0, 1/\alpha]$, $[1/\alpha, \beta]$, and $[\beta, \infty)$. The first of these integrals is estimated as

$$\int_0^{1/\alpha} (\alpha x)^r (\beta/x)^{-l} [L(\beta/x)]^\gamma \frac{dx}{x} = \alpha^r \beta^{-l} \int_0^{1/\alpha} x^{r+l} [L(\beta/x)]^\gamma \frac{dx}{x} = \alpha^r \beta^{-l} \int_0^1 (u/\alpha)^{r+l} [L(\alpha\beta/u)]^\gamma \frac{du}{u}$$
$$\le C(\alpha\beta)^{-l} \int_0^1 u^{r+l} \left([L(\alpha\beta)]^\gamma + [L(1/u)]^\gamma\right) \frac{du}{u} \le C(\alpha\beta)^{-l} [L(\alpha\beta)]^\gamma.$$

The integral over $[1/\alpha, \beta]$ is estimated as

$$\int_{1/\alpha}^{\beta} (\alpha x)^{-l} (\beta/x)^{-l} [L(\beta/x)]^\gamma \frac{dx}{x} = (\alpha\beta)^{-l} \int_{1/\alpha}^{\beta} [L(\beta/x)]^\gamma \frac{dx}{x} \le C(\alpha\beta)^{-l} [L(\alpha\beta)]^{\gamma+1}.$$

The third integral is estimated as

$$\int_\beta^\infty (\alpha x)^{-l} (\beta/x)^r [L(\beta/x)]^\gamma \frac{dx}{x} = \alpha^{-l} \beta^r \int_\beta^\infty x^{-r-l} [L(\beta/x)]^\gamma \frac{dx}{x}$$
$$= \alpha^{-l} \beta^r \int_1^\infty (\beta u)^{-r-l} [L(1/u)]^\gamma \frac{du}{u} \le C(\alpha\beta)^{-l}.$$

These three estimates prove (2.14) in the case $\alpha\beta > 1$, since $m_0(\alpha\beta) = (\alpha\beta)^{-l}$ in this case.

As we see from the proof of estimate (2.14), this estimate holds in the case where $r \ge 0$ (including $r = 0$) and $l = 2$ or $l = 3$ (no matter what r is). Therefore, estimate (2.16) holds because (2.16) is equivalent to (2.14) with r replaced by $r - 1$ $(r - 1 \ge 0)$. To prove (2.15), we note that

$$x^{-1} m_0(x) = x^{-1} \min(x^r, x^{-l}) = \min(x^{r-1}, x^{-l-1}) \le \min(x^{r-1}, x^{-l}) = m_1(x)$$

for $x > 0$. Therefore, (2.15) follows from (2.16). □

Applying this lemma, we can prove the following theorem which estimates the multiplier norms of the coefficients $\lambda(k, t)$ of the operator $W_r(t)$.

Theorem 2.3. *Let $r = 1, 2, \ldots$, $m = 0, 1, \ldots$, $V \subset \{1, \ldots, d\}$, and let μ satisfy the relation $|\mu| := |\mu_1| + \cdots + |\mu_d| = m$ for each c. Then, for $2^{-n-1} \le t \le 2^{-n}$, we have*

$$\sum_{k \in H(V,c,\rho(\mu))} |\Delta(V)\lambda(k,t)| \le C 2^{-r|m-n|} [|n-m|+1]^{d-1}, \quad 0 \le m \le n, \tag{2.20}$$

and

$$\sum_{k \in H(V,c,\rho(\mu))} |\Delta(V)\lambda(k,t)| \le C 2^{(-l+1)|m-n|} [|n-m|+1]^{d-1}, \quad m > n, \tag{2.21}$$

where C depends only on d and r.

Proof. We first assume that $V = \varnothing$ (i.e., $\ell = d$), so that $H(V, c, \rho(\mu))$ consists of at most one element $k \in \rho(\mu)$. Using (2.13), for this element k we obtain

$$|\Delta(V)\lambda(k,t)| = |\lambda(k,t)| \le C \int_0^\infty \cdots \int_0^\infty m_0(|k_d| s_1^{-1} \cdots s_{d-1}^{-1} t) \prod_{j=1}^{d-1} m_0(|k_j| s_j) \frac{ds_{d-1}}{s_{d-1}} \cdots \frac{ds_1}{s_1}. \tag{2.22}$$

If $\prod_{j=1}^d k_j = 0$, then the right side of (2.22) vanishes, and estimates (2.20) and (2.21) are trivial. Therefore, we can assume that $\prod_{j=1}^d k_j \ne 0$. Let us integrate the right side of (2.22) first with respect to s_{d-1}, and then s_{d-2}, and so on, and use (2.14) at each step. Then we obtain

$$|\Delta(V)\lambda(k,t)| \le C m_0(|k_1 \cdots k_d| t) [L(|k_1| \cdots |k_d| t)]^{d-1}.$$

Since $|k_1 \cdots k_d| \approx 2^m$, this provides the desired estimates (2.20), (2.21) in the case under consideration.

For the more general case when $V \neq \varnothing$ and $k \in H(V,c,\mu)$, we obtain from (2.13) the estimate

$$|\Delta(V)\lambda(k,t)| \leq C \int_0^\infty \cdots \int_0^\infty s_1^{-1} \cdots s_{d-1}^{-1} t\, m_1((|k_d|+1)s_1^{-1} \cdots s_{d-1}^{-1} t) \times \prod_{j=\ell+1}^{d-1} s_j m_1((|k_j|+1)s_j) \prod_{j=1}^{\ell} m_0(|k_j|s_j) \frac{ds_{d-1}}{s_{d-1}} \cdots \frac{ds_1}{s_1}. \tag{2.23}$$

We now make the integration with respect to s_{d-1} and use (2.16) to bound the resulting integral. We continue this procedure until we have integrated with respect to $s_{\ell+1}$. At this point, we have the estimate

$$|\Delta(V)\lambda(k,t)| \leq C \int_0^\infty \cdots \int_0^\infty s_1^{-1} \cdots s_\ell^{-1} t [L((|k_d|+1)\cdots(|k_{\ell+1}|+1)s_1^{-1}\cdots s_\ell^{-1} t)]^{d-\ell-1} \times m_1((|k_d|+1)\cdots(|k_{\ell+1}|+1)|s_1^{-1}\cdots s_\ell^{-1} t) \prod_{j=1}^{\ell} m_0(|k_j|s_j) \frac{ds_\ell}{s_\ell} \cdots \frac{ds_1}{s_1}. \tag{2.24}$$

From this point on, we integrate with respect to s_ℓ, $s_{\ell-1}$, and so on, using (2.15) at each step we. As a result, we have

$$|\Delta(V)\lambda(k,t)| \leq Ct|k_1 \cdots k_\ell| m_1((|k_1|+1)\cdots(|k_d|+1)t)[L((|k_1|+1)\cdots(|k_d|+1)t)]^{d-1}. \tag{2.25}$$

We continue the proof, assuming that $m \leq n$ for the moment. Since $m_1(x) = x^{r-1} = x^{-1}m_0(x)$, $0 \leq x \leq 1$, the right side of (2.25) is estimated as

$$\begin{aligned} &\leq C(|k_{\ell+1}|+1)^{-1}\cdots(|k_d|+1)^{-1} m_0(|k_1\cdots k_d|t)[L((|k_1|+1)\cdots(|k_d|+1)t)]^{d-1} \\ &\leq C(|k_{\ell+1}|+1)^{-1}\cdots(|k_d|+1)^{-1} 2^{-|m-n|r}[|m-n|+1]^{d-1} \end{aligned} \tag{2.26}$$

because $(|k_1|+1)\cdots(|k_d|+1)t \approx 2^{m-n} = 2^{-|m-n|}$.

Now, for each μ and c, any multi-integer $k \in H(V,c,\rho(\mu))$ satisfies the conditions $k \in \rho(\mu)$ and $k_1 = c_1, \ldots, k_\ell = c_\ell$. Since $[2^{\mu_j-1}] \leq |k_j| < 2^{\mu_j}$, $j = \ell+1, \ldots, d$, there are at most $2^{\mu_{\ell+1}+\cdots+\mu_d}$ such multi-integers k. On the other hand, $(|k_{\ell+1}|+1)^{-1}\cdots(|k_d|+1)^{-1} \leq 2^d 2^{-(\mu_{\ell+1}+\cdots+\mu_d)}$. Therefore, from (2.26) we obtain

$$\sum_{k \in H(V,c,\rho(\mu))} |\Delta(V)\lambda(k,t)| \leq C 2^{-|m-n|r}[|m-n|+1]^{d-1},$$

which proves (2.20).

For the case where $m > n$ in (2.21), the only difference in the proof is that the term m_1 in the right side of (2.25) is replaced by $[t(|k_1|+1)\cdots(|k_d|+1)]^{-l}$. In all other respects the proof is the same as for (2.20). □

We can use Theorem 2.3 to analyze the mapping properties of the operator $W_r(t)$. For this, we put

$$T_m := \sum_{\mu \in F_m} \delta_\mu, \quad F_m := \{\mu > 0 : |\mu| = m\}. \tag{2.27}$$

Here, in the definition of F_m the condition $\mu > 0$ means that $\mu_j > 0$, $j = 1, \ldots, d$, and $|\mu| = \mu_1 + \cdots + \mu_d$. As a consequence of Theorem 2.3 and the Marcinkiewicz multiplier theorem, we have the following statement.

Corollary 2.4. *Let $r = 1, 2, \ldots$, $m = 1, 2, \ldots$, and $n = 0, 1, \ldots$. Then, for $2^{-n-1} \le t \le 2^{-n}$, we have*

$$\|W_r(t)T_m f\|_p \le C \begin{cases} 2^{-r|n-m|}[|m-n|+1]^{d-1}\|T_m f\|_p & 0 < m \le n, \\ 2^{(-\ell+1)|n-m|}[|m-n|+1]^{d-1}\|T_m f\|_p & m > n, \end{cases} \tag{2.28}$$

where C depends only on r, d, and p.

Proof. We first note that $T_m^2 = T_m$ (i.e., T_m is a projector). Therefore, we need only verify the conditions of the Marcinkiewicz multiplier theorem (Theorem 2.1) for the operator $W_r(t)T_m$. The multiplier coefficients of this operator are equal to $\lambda(k,t)$ for $k \in \bigcup_{\mu \in F_m} \rho(\mu)$, and are zero otherwise. Therefore, condition $(*)$ of the Marcinkiewicz theorem follows from (2.20) and (2.21). □

We need another result concerning multipliers. Given a sequence $\alpha = (\alpha(k))_{k \in \mathbb{Z}^d}$ which itself is a bounded multiplier on $L_p(\mathbb{T}^d)$, we are interested in when the sequence of reciprocals $\alpha^{-1} := (1/\alpha(k))_{k \in \mathbb{Z}^d}$ is also a bounded multiplier on $L_p(\mathbb{T}^d)$.

We begin with a lemma which describes the effect of applying difference operators to products of sequences. If $\alpha := (\alpha(k))_{k\in\mathbb{Z}^d}$ and $\beta := (\beta(k))_{k \in \mathbb{Z}^d}$ are two sequences, then we denote by $\alpha\beta := (\alpha(k)\beta(k))_{k\in\mathbb{Z}^d}$ their coordinatewise product. We use our earlier notation (see (2.4) and (2.5)) for shift operators $\tau(V)$ and for difference operators $\Delta(V)$, $V \subset \{1, \ldots, d\}$.

We repeatedly use the following simple identity relating the differences of product sequences. For each $j = 1, \ldots, d$, we have

$$\Delta(\{j\})[\alpha\beta] = \tau(j)\alpha\tau(j)\beta - \alpha\beta = \Delta(\varnothing)\alpha\Delta(\{j\})\beta + \tau(j)\beta\Delta(\{j\})\alpha. \tag{2.29}$$

Here and in what follows $\Delta(\varnothing)$ is defined to be the identity operator.

Lemma 2.5. *If $V \subset \{1, \ldots, d\}$, then for any α and β we have*

$$\Delta(V)[\alpha\beta] = \sum_{V_0 \subset V} \Delta(V_0)\alpha\, \tau(V_0)\Delta(V \setminus V_0)\beta. \tag{2.30}$$

Proof. Because of the symmetry, it is enough to prove (2.30) for $V = \{1, \ldots, \ell\}$, $\ell = 1, \ldots, d$. The proof is carried out by induction on ℓ. For $\ell = 1$, from (2.29) we have

$$\Delta(\{1\})[\alpha\beta] = \Delta(\varnothing)\alpha\Delta(\{1\})\beta + \Delta(\{1\})\alpha\tau(\{1\})\beta$$

which is the desired identity (2.30) in the case $\ell = 1$.

Assume that formula (2.30) has been established for $V = \{1, \ldots, \ell\}$, and let $V' := \{1, \ldots, \ell+1\}$. We apply $\Delta(\{\ell+1\})$ to an arbitrary term of the sum (2.30) (for V). Using formula (2.29) with $j = \ell + 1$ for the product ab of the sequences $a = \Delta(V_0)\alpha$ and $b = \tau(V_0)\Delta(V \setminus V_0)\beta$, we obtain

$$\begin{aligned} \Delta(\{\ell+1\})[ab] &= \Delta(\varnothing)a\Delta(\{\ell+1\})b + \Delta(\{\ell+1\})a\tau(\ell+1)b \\ &= \Delta(V_0)\alpha\,\tau(V_0)\Delta(V' \setminus V_0)\beta + \Delta(V_0 \cup \{\ell+1\})\alpha\,\tau(\ell+1)\tau(V_0)\Delta(V \setminus V_0)\beta. \end{aligned} \tag{2.31}$$

Since $\Delta(V \setminus V_0) = \Delta(V' \setminus (V_0 \cup \{\ell+1\})$ and $\tau(\{\ell+1\})\tau(V_0) = \tau(V_0 \cup \{\ell+1\})$, we see that both terms in the right side of (2.31) appear in the sum (2.30) for V'. Note also that the terms in the right side of (2.31), obtained from different V_0's, are different. Finally, each term appearing in the right side of (2.30) for V' arises as a term in the right side of (2.31) for some V_0. Therefore, we have proved (2.30). □

Let X_n denote the linear subspace of $L_p(\mathbb{T}^d)$ spanned by the exponentials e_k, $k \in \bigcup_{\mu \in F_n} \rho(\mu)$. We show that the operator $W_r(t)$, $t \approx 2^{-n}$ is invertible on X_n.

Theorem 2.6. *Let $r, d = 1, 2, \ldots$, and $n = 0, 1, \ldots$. Then, for $2^{-n-1} \le t \le 2^{-n}$, the operator $W_r(t)$ is invertible on X_n, and the following estimate:*

$$\|[W_r(t)]^{-1} T_n f\|_p \le M_0 \|T_n f\|_p \tag{2.32}$$

is valid for $f \in L_p(\mathbb{T}^d)$, $1 < p < \infty$, where M_0 depends only on r, d, and p.

Proof. Let us show that the coefficients $\lambda^{-1}(k,t) := 1/\lambda(k,t)$, $k \in \bigcup_{\mu\in F_n} \rho(\mu)$, satisfy the conditions of the Marcinkiewicz multiplier theorem. We first bound $|\lambda(k,t)|$ from below by using the integral representation (2.9). Since $r + l$ is even, we see that $\prod_{j=1}^d \frac{(\sin k_j s_j/2)^{r+l}}{(k_j s_j/2)^l}$ does not change sign for $s_j > 0$ and therefore

$$|\lambda(k,t)| = \int_0^\infty \cdots \int_0^\infty |\eta(k, (s_1, s_2, \ldots, s_{d-1}, s_1^{-1} s_2^{-1} \cdots s_{d-1}^{-1} t))| \frac{ds_{d-1}}{s_{d-1}} \cdots \frac{ds_1}{s_1}.$$

For $s_j \in I_j := [|k_j|^{-1}, 2|k_j|^{-1}]$, the function $\frac{(\sin k_j s_j/2)^{r+l}}{(k_j s_j/2)^l}$ exceeds some constant $\ge C > 0$ dependent only on r. Then we have

$$2^{-d+1} |k_1 \cdots k_d| t \le k_d |s_1 \cdots s_{d-1}|^{-1} t \le |k_1 \cdots k_d| t$$

for $s_j \in I_j$, $j = 1, \ldots, d-1$, and

$$2^{-d-1} \le t |k_1 \cdots k_d| \le 1$$

for $k \in \bigcup_{\mu\in F_n} \rho(\mu)$.

This shows that the function $\left|\frac{[\sin(k_d s_1^{-1} \cdots s_{d-1}^{-1} t/2)]^{r+l}}{(k_d s_1^{-1} \cdots s_{d-1}^{-1} t/2)^l}\right|$ is also bounded from below on $I_1 \times \cdots \times I_{d-1}$. Hence, the function $|\eta(k, s_1, \ldots, s_{d-1}, s_1^{-1} \cdots s_{d-1}^{-1} t)|$ is bounded from below on $I_1 \times \cdots \times I_{d-1}$ as well. We find therefore

$$\begin{aligned} |\lambda(k,t)| &\ge C \int_{I_1} \cdots \int_{I_{d-1}} |\eta(k, (s_1, s_2, \ldots, s_{d-1}, s_1^{-1} s_2^{-1} \cdots s_{d-1}^{-1} t))| \frac{ds_{d-1}}{s_{d-1}} \cdots \frac{ds_1}{s_1} \\ &\ge C \int_{I_1} \cdots \int_{I_{d-1}} \frac{ds_{d-1}}{s_{d-1}} \cdots \frac{ds_1}{s_1} \ge C_1 > 0 \end{aligned} \tag{2.33}$$

with C_1 depending only on r and d. This shows that $(\lambda(k,t)^{-1})_{k\in\mathbb{Z}^d}$ satisfies the case $V = \varnothing$ of condition $(*)$ of the Marcinkiewicz theorem on $\cup_{\mu\in F_n} \rho(\mu)$ with M a constant depending on r and d.

We now show that condition $(*)$ of the Marcinkiewicz theorem is satisfied when $V \ne \varnothing$. However, first we prove by induction on the cardinality of V that if k and $\tau(V)k$ belong to F_n, then

$$|\Delta(V)\lambda(k,t)^{-1}| \le C \prod_{v\in V} |k_v|^{-1} \le C \prod_{v\in V} 2^{-\mu_v}, \tag{2.34}$$

where C is dependent only on r and d. We have already proved an estimate like (2.34) for the sequence λ (see (2.25), (2.26) for $m = n$). The case $V = \varnothing$ has already been proved. Assume that we have already proved (2.34) for an arbitrary set $V \subset \{1, \ldots, d\}$ of cardinality ℓ. Let us verify the validity of (2.34) for sets V of cardinality $(\ell + 1)$.

We use the identity (2.30) with $\alpha = \lambda(\cdot, t)$ and $\beta = \lambda(\cdot, t)^{-1}$. Then each coordinate of the sequence $\alpha\beta$ is equal to 1. Hence, the left side of (2.30) is equal to 0. This yields the relation

$$\alpha(k)\Delta(V)\beta(k) = -\sum_{V_0 \subset V,\ V_0 \ne \varnothing} \Delta(V_0)\alpha(k)\,\tau(V_0)\Delta(V \setminus V_0)\beta(k). \tag{2.35}$$

Now, if k and $\tau(V)k$ belong to $\bigcup_{\mu\in F_n} \rho(\mu)$, and $V_0 \subset V$, then $\tau(V_0)k$ and $\tau(V_0)\tau(V \setminus V_0)k = \tau(V)k$ also belongs to $\bigcup_{\mu\in F_n} \rho(\mu)$. Therefore, each term in the right side of (2.35) has already been estimated either by the induction hypothesis or due to the knowledge of $\alpha(k)$. Therefore, the V_0 term in the right side of (2.35) is estimated as

$$C \prod_{v \in V_0} |k_v|^{-1} \prod_{v \in V \setminus V_0} |k_v|^{-1} = C \prod_{v \in V} |k_v|^{-1}.$$

Dividing by $\alpha(k)$ in (2.35), and using (2.33) gives (2.34).

By the remarks made after the Marcinkiewicz theorem, we need only verify (*) for $\mu \in F_n$. For such μ, estimate (2.34) holds for each $k \in H(V, c, \rho(\mu))$. Since the cardinality of the set $H(V, c, \rho(\mu))$ does not exceed $\prod_{v \in V} 2^{\mu_v}$, we have verified (*). □

3. Direct and inverse theorems. In this section we prove direct and inverse theorems which compare the error $E'_{2^n}(f)_p$ of the approximation by polynomials with harmonics of hyperbolic cross with $\Omega_r(f, 2^{-n})_p$. In addition to the notation T_m introduced in (2.27), we define

$$T_0(f) := \sum_{F_0} \hat{f}(k) e_k, \quad F_0 := \{k : k_1 \dots k_d = 0\},$$

and

$$S_n(f) := \sum_{m=0}^{n} T_m(f). \tag{3.1}$$

It follows from the Littlewood–Paley theorem that the operators T_m, $m = 0, 1, \dots$, and S_n, $n = 0, 1, \dots$, are bounded in $L_p(\mathbb{T}^d)$, $1 < p < \infty$. Moreover, S_n is a projector such that $S_n(e_k) = e_k$ for each k satisfying the inequality $|k_1 \dots k_d| \leq 2^{n-d}$. Therefore, each function $f \in L_p(\mathbb{T}^d)$ can be expanded in the series

$$f = \sum_{m=0}^{\infty} T_m(f) \tag{3.2}$$

convergent in $L_p(\mathbb{T}^d)$, $1 < p < \infty$.

We begin with the following direct theorem.

Theorem 3.1. *Let $r = 1, 2, \dots$, and $1 < p < \infty$. Then, for each $f \in L_p(\mathbb{T}^d)$, we have*

$$E'_{2^n}(f)_p \leq C \sum_{m=n+1}^{\infty} \Omega_r(f, 2^{-m})_p, \quad n = 0, 1, \dots, \tag{3.3}$$

where the constant C depends only on r, d, and p.

Proof. For $f \in L_p(\mathbb{T}^d)$, the function $S_n(f)$ belongs to $\mathcal{T}'_{2^n}$. Therefore,

$$E'_{2^n}(f)_p \leq \sum_{m=n+1}^{\infty} \|T_m(f)\|_p. \tag{3.4}$$

We write $T_m(f) = [W_r(2^{-m})]^{-1} T_m W_r(2^{-m}) f$ (note that all of these operators commute), where $W_r(t)$ is an operator defined in (1.11). By Theorem 2.6, the operator $[W_r(2^{-m})]^{-1} T_m$ is bounded in $L_p(\mathbb{T}^d)$; therefore, from definition (1.12) of Ω_r we have

$$\|T_m(f)\|_p \leq C \|T_m W_r(2^{-m}) f\|_p \leq C \|W_r(2^{-m}) f\|_p = C \Omega_r(f, 2^{-m})_p. \quad □$$

We can also prove an accompanying theorem, inverse to Theorem 3.1. For the statement of this theorem, we define $E'_{2^{-1}}(f)_p := \|f\|_p$.

Theorem 3.2. *Let $r = 1,2,\ldots$, $n = 0,1,\ldots$, $1 < p < \infty$, and $2^{-n-1} \le t \le 2^{-n}$. Then, for each $f \in L_p(\mathbb{T}^d)$, we have*

$$\Omega_r(f,t)_p \le C \sum_{m=-1}^{n} 2^{-r|m-n|}[|m-n|+1]^{d-1} E'_{2^m}(f)_p, \tag{3.5}$$

where C depends at most on r, d, and p.

Proof. Let us write $f = \sum_{m=0}^{\infty} T_m(f)$ so that for $2^{-n-1} \le t \le 2^{-n}$ we have

$$W_r(t)f = \sum_{m=0}^{\infty} W_r(t)T_m(f).$$

As we have already noted, if $k_1 \cdots k_d = 0$, then the corresponding multiplier coefficient $\lambda(k,t)$ of $W_r(t)$ is equal to zero. Thus, $W_r(t)T_0(f) = 0$ and therefore we have

$$\Omega_r(f,t)_p = \|W_r(t)f\|_p \le \sum_{m=1}^{\infty} \|W_r(t)T_m(f)\|_p. \tag{3.6}$$

Estimate (2.28) of Corollary 2.4 yields

$$\|W_r(t)T_m(f)\|_p \le \|T_m(f)\|_p \begin{cases} 2^{-r|m-n|}[|m-n|+1]^{d-1} & 1 \le m \le n, \\ 2^{(-l+1)|m-n|}[|m-n|+1]^{d-1} & m > n. \end{cases} \tag{3.7}$$

Further, for $m = 1,2,\ldots$, we have

$$\|T_m(f)\|_p = \|f - S_m(f) - (f - S_{m-1}(f))\|_p \le \|f - S_m(f)\|_p + \|f - S_{m-1}(f)\|_p.$$

We have already proved that S_m is an L_p-bounded projector onto the space X_m spanned by all exponentials e_k with $k \in \cup_{m=0}^{n} \cup_{\mu \in F_m} \rho(\mu)$, and this space contains $T'_{2^{m-d}}$. Therefore,

$$\|f - S_m(f)\|_p \le C E'_{2^{m-d}}(f)_p, \quad m \ge d. \tag{3.8}$$

Moreover, as we have mentioned above, the operators $S_0, S_1, \ldots, S_{d-1}$ are bounded in $L_p(\mathbb{T}^d)$. Therefore,

$$\|f - S_m(f)\|_p \le C\|f\|_p, \quad m = 0,\ldots,d-1. \tag{3.9}$$

We use (3.8) and (3.9) to bound $\|T_m(f)\|_p$ in (3.7); then we use (3.6) to arrive at the following estimate:

$$\Omega_r(f,t)_p \le C \sum_{m=-1}^{n} 2^{-r|m-n|}[|m-n|+1]^{d-1} E'_{2^m}(f)_p + C \sum_{m=n+1}^{\infty} 2^{(-l+1)|m-n|}[|m-n|+1]^{d-1} E'_{2^m}(f)_p.$$

Since $E'_{2^m}(f)_p \le E'_{2^n}(f)_p$ for $m > n$, and $\sum_{m=n+1}^{\infty} 2^{(-l+1)|m-n|}[|m-n|+1]^{d-1} < \infty$ ($l \ge 2$), we see that the latter series above is bounded by $CE'_{2^n}(f)_p$. This term can be included in the first sum by increasing the constant C; this proves estimate (3.5). □

Corollary 3.3. *Let $r = 1,2,\ldots$, $1 < p < \infty$, and $0 \le t \le 1$. Then, for each $f \in L_p(\mathbb{T}^d)$, we have*

$$\Omega_r(f,t)_p \le C\|f\|_p,$$

where C depends on r, d, and p.

Proof. This estimate follows from (3.5), since $E'_{2^m}(f)_p \le \|f\|_p$. □

We can use the direct and inverse inequalities from Theorems 3.1-2 to prove Theorem 1.1. For this purpose we recall the discrete Hardy inequalities. For a sequence $a = (a_k)_{k\in\mathbb{Z}}$, we define

$$\|a\|_{\alpha,q} = (\sum_{k\in\mathbb{Z}}[2^{k\alpha}|a_k|]^q)^{1/q}$$

for $\alpha, q > 0$; the standard modification of this definition is assumed for $q = \infty$. The special case of the discrete Hardy inequality (see [9, p. 27]) asserts that if the sequences $a_k, b_k \geq 0$ satisfy the inequality

$$b_k \leq M_0 \sum_{j=k}^{\infty} a_j + M_0 2^{-kr} \sum_{j=-\infty}^{k} 2^{jr} a_j, \tag{3.10}$$

then for $0 < \alpha < r$ and $0 < q \leq \infty$ we have

$$\|b\|_{\alpha,q} \leq CM_0\|a\|_{\alpha,q}. \tag{3.11}$$

We need a slight improvement of this inequality. Namely, condition (3.10) can be changed to

$$b_k \leq M_0 \sum_{j=k}^{\infty} a_j + M_0 2^{-kr} \sum_{j=-\infty}^{k} [|k-j|+1]^{d-1} 2^{jr} a_j. \tag{3.12}$$

The proof that (3.12) implies (3.11) is similar to the proof that (3.10) implies (3.11) (see [9, p. 27]).

In the applications of the discrete Hardy inequalities that follow, we consider sequences that are not defined on the whole set $\mathbb{Z}$. To apply this inequality, we simply complete the definition of a sequence with zero for such values of $k \in \mathbb{Z}$ on which the sequence is not defined.

Proof of Theorem 1.1. Let parameters $\alpha > 0$ and $q > 0$ be fixed. We discuss only the case $q < \infty$ (a simple modification treats the case $q = \infty$). Suppose that expression (1.13) is finite and does not exceed M. Then from Theorem 3.1 and the discrete Hardy inequalities with $a_j = \Omega_r(f, 2^{-j-1})_p$, $j \geq 0$, and with $b_j = E'_{2^j}(f)_p$, $j \geq 0$, it follows that

$$\left(\sum_{k=0}^{\infty}[2^{k\alpha}E'_{2^k}(f)_p]^q\right)^{1/q} \leq CM$$

which is one of the implications of Theorem 1.1.

If (1.1) with $E_{2^k}(f)_p$ replaced by $E'_{2^k}(f)_p$ is finite and does not exceed M, then (3.5) shows that (3.12) holds for $b_j = \Omega_r(f, 2^{-j})_p$, $j \geq 0$, and $a_j = E'_{2^j}(f)_p$, $j \geq -1$. Hence, the discrete Hardy inequality yields

$$\left(\sum_{k=0}^{\infty}[2^{k\alpha}\Omega_r(f,2^{-k})_p]^q\right)^{1/q} \leq C\left(\sum_{k=-1}^{\infty}[2^{k\alpha}E'_{2^k}(f)_p]^q\right)^{1/q} \leq C(M + \|f\|_p),$$

since by definition $E'_{2^{-1}}(f)_p = \|f\|_p$. □

4. Characterization of $\mathcal{A}_q^\alpha(L_p(\mathbb{T}^d), \mathcal{T}_n)$. In this section, we show how to extend the results of Sec. 3 to the case of approximation by the elements of $\mathcal{T}_n$. For this purpose, we expand functions of $L_p(\mathbb{T}^d)$ in the sum of functions of fewer variables. We begin by defining certain operators on $L_p(\mathbb{T}^d)$ that associate to a function f a new function of possibly fewer variables. For $j = 1, \ldots, d$, we put

$$Y(j)f := \frac{1}{2\pi}\int_{\mathbb{T}} f(x)dx_j.$$

Then $Y(j)f$ is a function of x_i, $i \neq j$. More generally, for any set $V \subset \{1, \dots, d\}$ we define

$$Y(V) := \prod_{j \in V} Y(j),$$

and

$$Y'(V) := \prod_{j \in V} (I - Y(j)),$$

where I is the identity operator. For a subset $V \subset \{1, \dots, d\}$, we denote by V' its complement $\{1, \dots, d\} \setminus V$. Then we define the operator

$$\Psi(V) := Y'(V)Y(V').$$

We need some properties of these operators. Let $L_p(\mathbb{T}^V)$ denote the set of 2π-periodic functions of x_j, $j \in V$, that belong to L_p with respect to these variables, and let $L_p^0(\mathbb{T}^V)$ denote the set of functions of $L_p(\mathbb{T}^V)$ that satisfy the condition

$$Y(j)f = 0, \quad j \in V.$$

If $g \in L_p(\mathbb{T}^V)$, then $g \in L_p(\mathbb{T}^d)$ and

$$\|g\|_{L_p(\mathbb{T}^d)} = \|g\|_{L_p(\mathbb{T}^V)}.$$

The following lemma summarizes the properties of the operators $\Psi(V)$.

Lemma 4.1. *The operators $\Psi(V)$ have the following properties:*

(i) *$\Psi(V)e_k = e_k$, if $k_j = 0$, for all $j \notin V$, and if $\prod_{j \in V} k_j \neq 0$. For all other k, we have $\Psi(V)e_k = 0$;*

(ii) *$\Psi(V)$ is a projector from $L_p(\mathbb{T}^d)$ onto $L_p^0(\mathbb{T}^V)$ with the norm not exceeding $2^{|V|}$;*

(iii) *for each $f \in L_p(\mathbb{T}^d)$, we have*

$$f = \sum_{V \subset \{1, \dots, d\}} f_V, \quad \text{where } f_V := \Psi(V)f \in L_p^0(\mathbb{T}^V). \tag{4.1}$$

Proof. Property (i) is trivial. Let us prove property (ii). From Hölder's inequality it follows that $Y(V)$ maps $L_p(\mathbb{T}^d)$ into $L_p(\mathbb{T}^{V'})$ with norm ≤ 1. Hence $I - Y(j)$ has norm ≤ 2, which gives our claim about the norm of $\Psi(V)$. To check the range of $Y(V)$, it is enough to check the exponentials e_k, $k \in \mathbb{Z}^d$, since their linear span is dense in $L_p(\mathbb{T}^d)$. By property (i), each exponential e_k is mapped into the closed space $L_p^0(\mathbb{T}^V)$. Moreover, the linear span of the exponentials preserved by $\Psi(V)$ is dense in $L_p^0(\mathbb{T}^V)$.

Property (iii) is a consequence of the identity

$$I = \prod_{j=1}^{d} (I - Y(j) + Y(j)). \quad \square$$

We can now extend the results of Sec. 3 to the approximation by elements of $\mathcal{T}_n$. For $V \subset \{1, \dots, d\}$, we denote by $W_r(t, V)$ the operator W_r of (1.11), applied to the variables x_j, $j \in V$. For $f \in L_p(\mathbb{T}^d)$, we define

$$\Omega_r(f, t, V)_p := \|W_r(t, V)f_V\|_{L_p(\mathbb{T}^V)}, \tag{4.2}$$

where f_V is the function given by (4.1); and let

$$\Omega_r^*(f, t)_p := \sum_{V \subset \{1, \dots, d\}} \Omega_r(f, t, V)_p. \tag{4.3}$$

We denote by $\Gamma_n(V)$ and $\Gamma_n'(V)$ the frequencies k from Γ_n and Γ_n' restricted to V. The spaces $\mathcal{T}(\Gamma_n(V))$ and $\mathcal{T}(\Gamma_n'(V))$ are the closure of the linear span of the exponentials e_k with frequencies from $\Gamma_n(V)$

and $\Gamma'_n(V)$, respectively. For $f \in L_p(\mathbb{T}^V)$, we denote by $E_n(f, V)_p$ and $E'_n(f, V)_p$ the error of $L_p(\mathbb{T}^V)$-approximation from the spaces $\mathcal{T}(\Gamma_n(V))$ and $\mathcal{T}(\Gamma'_n(V))$, respectively. Then, for $g \in L^0_p(\mathbb{T}^V)$, we have

$$E_n(g)_p \leq 2^{|V|} E'_n(g, V)_p. \tag{4.4}$$

In fact, if T is an element of $\mathcal{T}(\Gamma'_n(V))$, then, by Lemma 4.1, $\Psi(V)T \in \mathcal{T}_n$ and also $\Psi(V)g = g$, since $\Psi(V)$ is a projector on $L^0_p(\mathbb{T}^V)$. Hence,

$$\|g - \Psi(V)T\|_{L_p(\mathbb{T}^d)} = \|\Psi(V)(g-T)\|_{L_p(\mathbb{T}^V)} \leq \|\Psi(V)\| \; \|g - T\|_{L_p(\mathbb{T}^V)}.$$

Taking an infimum over all $T \in \mathcal{T}(\Gamma'_n(V))$, and using the fact that $\|\Psi(V)\| \leq 2^{|V|}$, we obtain (4.4).

The function f_V belongs to $L^0_p(\mathbb{T}^V)$. Hence, it follows from Theorem 3.1 (applied to the variables in V) and (4.4) that

$$E_{2^n}(f_V)_p \leq C \sum_{m=n+1}^{\infty} \Omega_r(f_V, 2^{-m}, V)_p, \quad n = d, d+1, \ldots. \tag{4.5}$$

Theorem 4.2. *Let $r = 1, 2, \ldots$, $1 < p < \infty$, and let $f \in L_p(\mathbb{T}^d)$. Then we have*

$$E_{2^n}(f)_p \leq C \sum_{m=n+1}^{\infty} \Omega^*_r(f, 2^{-m})_p, \quad n = 0, 1, \ldots, \tag{4.6}$$

where the constant C depends only on r, d, and p.

Proof. The proof follows by writing $f = \sum_{V \subset \{1,\ldots,d\}} f_V$ and using (4.5) on each term. □

We can also prove an inverse theorem for the approximation by elements of $\mathcal{T}_n$, which is an analog of Theorem 3.2. For this, we note that if $f \in L_p(\mathbb{T}^d)$, then

$$E'_n(f_V, V)_p \leq 2^{|V|} E_n(f)_p. \tag{4.7}$$

In fact, if $T \in \mathcal{T}_n$, then $\Psi(V)T \in \mathcal{T}(\Gamma'_n(V))$ and

$$\|f_V - \Psi(V)T\|_{L_p(\mathbb{T}^d)} = \|\Psi(V)(f-T)\|_{L_p(\mathbb{T}^V)} \leq 2^{|V|} \|f - T\|_{L_p(\mathbb{T}^d)}.$$

Therefore, taking an infimum over all $T \in \mathcal{T}_n$, we arrive at (4.7).

Theorem 4.3. *Let $r = 1, 2, \ldots$, $n = 0, 1, \ldots$, $1 < p < \infty$, and let $2^{-n-1} \leq t \leq 2^{-n}$. Then for each function $f \in L_p(\mathbb{T}^d)$ we have*

$$\Omega^*_r(f, t)_p \leq \; C \sum_{m=-1}^{n} 2^{-r|m-n|} [|m-n| + 1]^{d-1} E_{2^m}(f)_p, \tag{4.8}$$

where the constant C depends at most on r, d, and p.

Proof. For each $V \subset \{1, \ldots, d\}$, from Theorem 3.2 we have

$$\Omega_r(f_V, t, V)_p \leq \; C \sum_{m=-1}^{n} 2^{-r|m-n|} [|m-n| + 1]^{d-1} E'_{2^m}(f_V, V)_p. \tag{4.9}$$

By (4.7), we can replace $E'_{2^m}(f_V, V)_p$ by $E_{2^m}(f)_p$. Therefore, taking a sum in (4.9) over all $V \subset \{1, \ldots, d\}$, and using the definition of Ω^*_r, we obtain (4.8). □

As a consequence of Theorems 4.2 and 4.3, we have the following characterization of the approximation spaces $\mathcal{A}^\alpha_q(L_p(\mathcal{T}_n))$.

Theorem 4.4. *Let $r = 1, 2, \ldots$, for $1 < p < \infty$, $0 < q \le \infty$, $0 < \alpha < r$. Then $f \in A_q^\alpha(L_p(\mathbb{T}^d), (\mathcal{T}_n))$ if and only if the quantity*

$$\begin{cases} \left(\sum_{k=0}^{\infty}[2^{k\alpha}\Omega_r^*(f, 2^{-k})_p]^q\right)^{1/q} & 0 < q < \infty, \\ \sup_{0 \le k < \infty} 2^{k\alpha}\Omega_r^*(f, 2^{-k})_p & q = \infty, \end{cases} \tag{4.10}$$

is finite. Moreover, expression (1.13) *after adding $\|f\|_p$ is equivalent to $\|f\|_{A_q^\alpha(L_p(\mathbb{T}^d),(\mathcal{T}_n))}$.*

Proof. The proof is carried out simply in the same way as for Theorem 1.1, using the Hardy inequality together with the direct and inverse estimates from Theorems 4.2, 4.3. □

5. Some alternate moduli of smoothness. In this section we mention (without proofs) certain modifications of $\Omega_r(f, t)_p$, which could also be used in the direct and inverse theorems of the previous two sections.

5.1. Bounded integration. In the definition (1.11) of $W_r(t)$ we integrate with respect to the variables s_j over the interval $(0, \infty)$. This choice seems to be the most convenient notationally. We could equally as well define

$$W_r(t)(f) := \int_t^1 \int_{s_1^{-1}t}^1 \cdots \int_{s_1^{-1}\ldots s_{d-2}^{-1}t}^1 w_r(s_1, s_2, \ldots, s_{d-1}, s_1^{-1}s_2^{-1}\cdots s_{d-1}^{-1}t)(f)\frac{ds_{d-1}}{s_{d-1}}\cdots\frac{ds_1}{s_1}$$

with $w_r(s) := A_2(s)\tilde{\Delta}^r(s)$, and use this in the definition of $\Omega_r(f, t)_p$ for $r = 1, 2, \ldots$, $0 < t \le \frac{1}{2}$, and $1 < p < \infty$. Note that in the above-mentioned definition we can take $l = 2$ for all values $r = 1, 2, \ldots$, since s_j are bounded.

5.2. The case $p = 2$. In the case of the approximation in $L_2(\mathbb{T}^d)$, several simplifications can be made in the definition of $\Omega_r(f, t)_2$. For example, $\Omega_r(f, t)_2$ can be replaced either by

$$\left(\int_0^\infty \cdots \int_0^\infty \|w_r(s_1, s_2, \ldots, s_{d-1}, s_1^{-1}s_2^{-1}\cdots s_{d-1}^{-1}t)(f)\|_2^2 \frac{ds_{d-1}}{s_{d-1}}\cdots\frac{ds_1}{s_1}\right)^{\frac{1}{2}} \tag{5.1}$$

with $w_r(s) := A_1(s)\tilde{\Delta}^r(s)$ ($l = 1$ in this case) or by

$$\left(\int_0^\infty \cdots \int_0^\infty \|\tilde{\Delta}^r(s_1, s_2, \ldots, s_{d-1}, s_1^{-1}s_2^{-1}\cdots s_{d-1}^{-1}t)(f)\|_2^2 \frac{ds_{d-1}}{s_{d-1}}\cdots\frac{ds_1}{s_1}\right)^{\frac{1}{2}} \tag{5.2}$$

for $r = 1, 2, \ldots$ and $t > 0$. We can also modify definitions (5.1) and (5.2) by integrating with finite limits as in Sec. 5.1.

One can prove direct and inverse theorems analogous to those in Secs. 3, 4 for the $L_2(\mathbb{T}^d)$-approximation by hyperbolic trigonometric polynomials, using the above moduli. The proofs rely on the Fourier series expansion of functions in $L_2(\mathbb{T}^d)$. We state below the theorems that correspond to redefining $\Omega_r(f, t)_2$ by (5.2).

Theorem 5.1. *Let $r = 1, 2, \ldots$. Then for each $f \in L_2(\mathbb{T}^d)$ we have*

$$E_n'(f)_2 \le C\Omega_r(f, n^{-1})_2, \quad n = 1, 2, \ldots$$

with $\Omega_r(f, t)_2$ defined by (5.2) *and the constant C dependent only on r and d.*

Theorem 5.2. *Let $r = 1, 2, \ldots$, $n = 0, 1, \ldots$, and $2^{-n-1} \le t \le 2^{-n}$. Then for each $f \in L_2(\mathbb{T}^d)$ we have*

$$\Omega_r(f, t)_2 \le C\left(\sum_{m=-1}^{n} 2^{-2r|m-n|}[|m-n|+1]^{d-1}E_{2^m}'(f)_2^2 + \sum_{m=n+1}^{\infty}[|m-n|+1]^{d-1}E_{2^m}'(f)_2^2\right)^{\frac{1}{2}} \tag{5.3}$$

with $\Omega_r(f, t)_2$ defined by (5.2) *and C dependent on r and d.*

The appearance of the second sum in the right side of (5.3) is related to the fact that $\Omega_r(f, t)_2$ is not bounded by $C\|f\|_2$ (with C independent of t) in the case of using definition (5.2). The direct and inverse estimates with $\Omega_r(f, t)_2$ defined by (5.1) are the same as (3.3) and (3.5).

5.3. Discretized moduli in $L_2(\mathbb{T}^d)$. Let us give another form for moduli of smoothness, which is useful for the approximation by polynomials with harmonics of hyperbolic crosses. The starting point is the mixed modulus of smoothness defined for $f \in L_2(\mathbb{T}^d)$ and a vector $(t_1, \ldots, t_d)$ with $t_j \geq 0$, $j = 1, \ldots, d$, by

$$\Omega^r(f, (t_1, \ldots, t_d))_2 := \sup_{|s_j| \leq t_j} \|\Delta^r(s_1, \ldots, s_d) f\|_2$$

where $\Delta^r(s_1, \ldots, s_d)f$ is the rth difference of f, defined as in (1.7) but with ordinary differences replacing the symmetric differences. For the mixed modulus the following theorem holds.

Theorem 5.3. *([10]) Let $r = 1, 2, \ldots$. Then for each $f \in L_2(\mathbb{T}^d)$ we have*

$$E'_{2^n}(f)_2 \leq C\Big(\sum_{|\mu|=n} \Omega^r(f, (2^{-\mu_1}, \ldots, 2^{-\mu_d})_2^2\Big)^{\frac{1}{2}}, \quad n = 0, 1, \ldots,$$

where the constant C depends only on r and d, and the summation is taken over all $\mu \in \mathbb{Z}^d$ such that $\mu \geq 0$ and $|\mu| := |\mu_1| + \cdots + |\mu_d| = n$.

This theorem, as well as the following inverse theorem, can be proved using a multiplier theorem similar to our consideration in Secs. 3, 4.

Theorem 5.4. *Let $r = 1, 2, \ldots$ and $n = 0, 1, \ldots$. Then for each $f \in L_2(\mathbb{T}^d)$ we have*

$$\sum_{|\mu|=n} \Omega^r(f, (2^{-\mu_1}, \ldots, 2^{-\mu_d}))_2^2$$
$$\leq C\Big(\sum_{m=-1}^{n} 2^{-2r|m-n|}[|m-n|+1]^{d-1} E'_{2^m}(f)_2^2 + \sum_{m=n+1}^{\infty} [|m-n|+1]^{d-1} E'_{2^m}(f)_2^2\Big),$$

where the constant C depends on r and d.

Using these two theorems, we can characterize the classes $\mathcal{A}_q^\alpha(L_2(\mathbb{T}^d), (T'_n))$ in the same way as in Theorem 1.1.

5.4. Further remarks. The advantage of these alternative moduli is two-fold. On the one hand, it is sometimes possible to simplify the direct theorem of approximation. For example, in Theorem 5.1 the right side contains one term instead of the sum, as in Theorem 3.1.

Information about one of the moduli of smoothness gives information on the others. For example, if for a function $f \in L_2^0(\mathbb{T}^d)$ one of the moduli satisfies the inequality

$$\Omega_r(f, t)_2 \leq Ct^\alpha |\ln t|^\beta \tag{5.4}$$

for $0 < \alpha < r$, $\beta \geq 0$, then by Theorems 3.1, 3.2 (and their analogs such as Theorems 5.1, 5.2), relation (5.4) is valid for the other moduli of this subsection as well. Similarly, (5.4) is equivalent to

$$\Big(\sum_{|\mu|=n} \Omega^r(f, (2^{-\mu_1}, \ldots, 2^{-\mu_d})_2^2\Big)^{1/2} \leq C2^{-\alpha n} n^\beta.$$

For a given function f it is sometimes simpler to estimate these moduli than our modulus (1.12). For example, suppose that $f(x) = \prod_{j=1}^d f_j(x_j)$ with $f_j \in L_2^0(\mathbb{T})$, $j = 1, \ldots, d$, and suppose that the univariate modulus of smoothness $\omega_r(f_j, h)_2$ of f_j satisfies the condition

$$\omega_r(f_j, h)_2 \leq h^\alpha |\ln h|^\beta, \quad h > 0, \quad j = 1, \ldots, d,$$

for some $\alpha > 0$, $\beta \geq 0$. Then it is easy to verify that

$$\left(\sum_{|\mu|=n} \Omega^r(f,(2^{-\mu_1},\dots,2^{-\mu_d})_2^2 \right)^{1/2} \leq C2^{-\alpha n} n^{d\beta+\frac{d-1}{2}}.$$

Using our previous remarks, we can derive from this that the modulus of (1.12) (with $p = 2$) satisfies the inequality

$$\Omega_r(f,t)_2 \leq Ct^{\alpha}|\ln t|^{d\beta+\frac{d-1}{2}}.$$

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