Besov Regularity for 2-D Elliptic Boundary Value Problems with Variable Coefficients

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Abstract

This paper is concerned with some theoretical foundations for adaptive methods for elliptic boundary value problems. The approximation order that can be achieved by an adaptive method is determined by the Besov regularity of the weak solution. We study this problem for second order elliptic problems in $\mathbf{R}^2$. The investigations are based on intermediate Schauder estimates and on some potential theoretic framework.

Key Words: Besov spaces, elliptic boundary value problems, Schauder estimates, adaptive methods, nonlinear approximation, wavelets

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1 Introduction

This paper is concerned with boundary value problems of the form

$$\begin{aligned} L(u) &= f \quad \text{on} \quad \Omega \subset \mathbf{R}^2, \quad f \in L_2(\Omega), \\ B(u) &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where L is a second order elliptic differential operator of the form

$$L = \sum_{|\beta| \leq 2} p_\beta(x) D^\beta \tag{1.2}$$

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and B expresses the boundary conditions. We assume that the domain Ω is at least **minimally smooth** in the sense of Stein [S]. We will consider (1.1) in the weak formulation

$$a(u,v) = (f,v) \quad \text{for all} \quad v \in H^1_B(\Omega), \tag{1.3}$$

where $H^1_B(\Omega)$ is a suitable subspace of $H^1(\Omega) = W^1(L^2(\Omega))$ depending on the boundary conditions. Since the problem is assumed to be elliptic, the **energy inner product** $a(\cdot,\cdot)$ is equivalent to the Sobolev norm on H^1, i.e.,

$$a(\cdot,\cdot) \sim \|\cdot\|^2_{H^1(\Omega)}.$$

In this paper, '$a \sim b$' means that both quantities can be uniformly bounded by some constant multiple of each other. Likewise, ' $\lesssim$ ' indicates inequality up to constant factors. The numerical treatment of (1.3) is in general performed by means of a **Galerkin** approach, i.e., we consider a nested sequence $\{S_j\}_{j\geq 0}$ of finite dimensional subspaces of H^1_B whose union is dense in H^1_B and project (1.3) onto the spaces S_j. Therefore, we have to solve the problems

$$a(u_j,v) = (f,v), \quad v \in S_j, \tag{1.4}$$

for $u_j \in S_j$, which corresponds to solving finite-dimensional linear systems. One widespread strategy to define the spaces $\{S_j\}_{j\geq 0}$ is to use **finite elements** associated with a given triangulation of the domain Ω. Another method which we focus on here is to use some kind of (orthogonal or biorthogonal) **wavelet basis.**

We are interested in the approximation order provided by the Galerkin scheme. This question is clearly related with the smoothness of the solution u in (1.3). If the domain is sufficiently smooth then the weak solution is in fact in $W^2(L_2(\Omega))$, see Wloka [W] for details. In this case, a Galerkin method obtained by **uniform grid refinement** is sufficient in the sense that on $\mathbf{R}^d$ we can expect estimates of the form

$$\|u-u_j\|_{L_2(\Omega)} \leq C2^{-2j}|u|_{W^2(L_2(\Omega))}, \quad \text{i.e.,} \quad \|u-u_j\|_{L_2(\Omega)} = O(n_j^{-2/d}), \quad u_j \in S_j, \tag{1.5}$$

where $|\cdot|_{W^2(L_2(\Omega))}$ denotes the corresponding Sobolev seminorm and n_j denotes the dimension of S_j. We refer to such methods as **linear** since the whole linear space S_j is used for the approximation. An estimate of the form (1.5) does no longer hold for nonsmooth domains, e.g. for domains with edges and corners, for then the smoothness of u in the scale of Sobolev spaces decreases significantly. For more specific information concerning Sobolev regularity for general domains, the reader is referred e.g. to Grisvard [G] or Kondrat'ev and Oleinik [KO]. In the worst case, the solution u will be only in $W^1(L_2(\Omega))$. As a consequence, the order of convergence for linear methods drops down. One way to overcome this difficulty is to use **adaptive methods.** In this case, the underlying grid is refined only in regions where the approximation u_j is still "far away" from the exact solution u.

In general, an adaptive method can be interpreted as some kind of **non-linear approximation.** The general setting can be described as follows. Let $\mathcal{M}_n$ denote a non-linear manifold of dimension n and let X be some topological space. Then we are interested in the error of approximation of $f \in X$ by $\mathcal{M}_n$,

$$\sigma_n(f)_X := \sigma(f, \mathcal{M}_n)_X := \inf_{a \in \mathbf{R}^n} \|f - \mathcal{M}(a)\|_X \tag{1.6}$$

and we want to describe the functions which have a specific order of approximation, i.e.,

$$\sigma_n(f)_X = O(n^{-\alpha}), \quad \alpha > 0. \tag{1.7}$$

In our case, the nonlinear manifold $\mathcal{M}_n$ will always be defined with respect to a suitable wavelet bases which is defined as follows. Let for m sufficiently large D_m denote the corresponding univariate Daubechies wavelet and let $\phi = \phi_m$ be the Daubechies scaling function associated with $\psi = D_m$, see [D]. We set $\psi^0 = \phi$, $\psi^1 = \psi$ and consider the set

$$\Psi := \{\psi^e\}_{e \in E}, \quad \psi^e(x_1, x_2) := \psi^{e_1}(x_1)\psi^{e_2}(x_2), \ e \in E, \tag{1.8}$$

where E denotes the set of nontrivial vertices in the unit cube. Let

$$\mathcal{D} := \{ \ I \ | I = 2^{-j}k + 2^{-j}[0,1]^2 \} \tag{1.9}$$

be the set of all dyadic cubes in $\mathbf{R}^2$. Then the functions

$$\eta_I := \eta_{j,k} = 2^j \eta(2^j \cdot - k), \quad I = 2^{-j}k + 2^{-j}[0,1]^2, \quad k \in \mathbf{Z}^2, j \in \mathbf{Z}, \eta \in \Psi \tag{1.10}$$

form an orthonormal basis for $L_2(\mathbf{R}^2)$. We want to approximate the solution u of (1.1) by linear combinations of n of the basis elements η_I, i.e., in our case the nonlinear manifold $\mathcal{M}_n$ consists of all functions

$$S = \sum_{(I,\eta) \in \Lambda} a_{I,\eta} \eta_I \tag{1.11}$$

with $\Lambda \in \mathcal{D} \times \Psi$ of cardinality n, and we consider the error

$$\sigma_n(u)_{L_2(\Omega)} = \inf_{S \in \mathcal{M}_n} \|u - S\|_{L_2(\Omega)}. \tag{1.12}$$

Then the aim is to establish an estimate for σ_n which is similar to (1.5). When dealing with wavelets, the spaces of functions that can be approximated by a certain order are well-known, see DeVore, Jawerth and Popov [DJP]. The main result is

$$\sum_{n=1}^{\infty} [n^{\alpha/d} \sigma_n(F)_{L_2(\Omega)}]^\tau \frac{1}{n} < \infty \iff F \in B^\alpha, \tag{1.13}$$

where $B^\alpha := B^\alpha_\tau(L_\tau)$, $\tau := (\frac{\alpha}{d} + \frac{1}{2})^{-1}$ are the **Besov** spaces. In general, a Besov space $B^\alpha_q(L_p)$ is defined as the space of all functions F for which

$$|F|_{B^\alpha_q(L_p)} := \begin{cases} (\int_0^\infty [t^{-\alpha}\omega_r(F,t)_p]^q dt/t)^{1/q}, & 0 < q < \infty, \\ \sup_{t \geq 0} t^{-\alpha}\omega_r(F,t)_p, & q = \infty, \end{cases} \tag{1.14}$$

$0 < \alpha < r$, is finite. Clearly, $\omega_r(F,t)$ denotes the modulus of smoothness of order r. (1.14) is a semi-norm for $B^\alpha_q(L_p)$. If we add $\|F\|_{L_p(\Omega)}$, we obtain a norm for $B^\alpha_q(L_p)$. it is well-known that different values of $r > \alpha$ give equivalent norms, see DeVore and Popov [DP]. Consequently, to make sure that we can achieve a certain approximation order, we have to show that the solution u is in the corresponding Besov space. Especially, to obtain an estimate similar to (1.5) in $\mathbf{R}^2$ we have to check that u is contained in the scale of spaces $B^\alpha_\tau(L_\tau)$, $\tau = (\frac{\alpha}{2} + \frac{1}{2})^{-1}$, $0 < \alpha < 2$. This is the main objective of this paper.

To prove the Besov regularity, we will use the fact that Besov spaces can be characterized by certain sequence norms, i.e., one has the following characterization.

Proposition 1.1 *Let ϕ and ψ be in $C^r(\mathbf{R})$. Furthermore, let $\mathcal{D}^+$ be the set of all dyadic cubes with measure < 1 and let P_0 denote the orthogonal projection onto $S_0 = span\{\phi(\cdot - k_1)\phi(\cdot - k_2),\ k = (k_1, k_2) \in \mathbf{Z}^2\}$. If $r > \alpha > 0$ and $\tau = (\alpha/2 + 1/2)^{-1}$, then a function F is in the Besov space $B^\alpha_\tau(L_\tau(\mathbf{R}^2))$, if and only if,*

$$F = P_0(F) + \sum_{I \in \mathcal{D}^+} \sum_{\eta \in \Psi} \langle F, \eta_I \rangle \eta_I \tag{1.15}$$

with

$$\|P_0(F)\|_{L_\tau(\mathbf{R}^2)} + \left(\sum_{I \in \mathcal{D}^+} \sum_{\eta \in \Psi} |\langle F, \eta_I \rangle|^\tau \right)^{1/\tau} < \infty \tag{1.16}$$

and (1.16) *provides an equivalent (quasi-)norm for $B^\alpha_\tau(L_\tau(\mathbf{R}^2))$.*

According to Proposition 1.1 we have to estimate the wavelet coefficients of the solution u.

For two model problems, i.e., for Laplace's equation:

$$\begin{aligned} -\Delta u &= f \quad \text{on} \quad \Omega \subset \mathbf{R}^d, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.17}$$

and for the Dirichlet problem for harmonic functions on Ω:

$$\begin{aligned} \Delta v &= 0 \quad \text{on} \quad \Omega \subset \mathbf{R}^d, \\ v &= g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.18}$$

these questions have been studied in [DD] and a complete answer was given for arbitrary dimensions. It turned out that the index α can be chosen very large. The results were based on some recent results of Jerison and Kenig [JK] and on some specific properties of harmonic Besov spaces. However, these specific properties do not carry over to general Besov spaces, so that different methods are needed for more general operators.

We will discuss two different approaches. The first one is based on the intermediate Schauder estimates introduced by Gilbarg and Hörmander [GH]. Intermediate Schauder estimates are generalizations of the classical Schauder estimates for smooth domains. For a Lipschitz domain Ω, classical Schauder estimates still hold strictly in the interior of Ω, but one clearly looses regularity as one approaches the boundary. Intermediate Schauder estimates measure this lack of regularity in terms of the distance to the boundary. Smoothness estimates for the solution u in terms of intermediate Schauder norms were derived by Gilbarg and Hörmander [GH], and they can be used to estimate wavelet coefficients and to establish the Besov regularity of u.

The second approach uses some kind of potential theoretic techniques. We assume that the solution u has an integral expression with respect to some function g on the boundary $\partial\Omega$ and some Kernel $K(x,q)$ on $\Omega \times \partial\Omega$. Then the decay of the derivatives of $K(x,q)$ directly determines the Besov regularity of u. Observe that for some important special case, e.g., for the classical Dirichlet problem, such an integral expression is always available since one can represent the solution in terms of layer potentials, see e.g. Kenig [K] for details.

2 Schauder Estimates and Besov Spaces

To estimate the Besov regularity of the solution u of (1.1) it is convenient to introduce a norm that measures how the smoothness of u decreases as we approach the boundary. It turns out that a suitable family of norms is provided by the intermediate Schauder estimates introduced by Gilbarg and Hörmander [GH].

If $N < a < N+1$ for a non-negative integer N then the Hölder spaces Lip_a are defined to be the set of N times continuously differentiable functions in Ω such that

$$|u|_a = \sum_{|\beta|\le N} |D^\beta u|_0 + \sum_{|\beta|=N} \sup_{x,y\in\Omega} |D^\beta u(x) - D^\beta u(y)||x-y|^{N-a} < \infty, \quad |u|_0 := \sup_{x\in\Omega} |u(x)|.$$

Let Ω_r be the set of points x in Ω with distance $\delta(x) := \inf_{q\in\partial\Omega} |x-q|$ greater than r to $\partial\Omega$. Then $\mathrm{Lip}_a^{(b)}(\Omega)$ is defined as the set of all functions u on Ω which restrict to a function in $\mathrm{Lip}_a(\Omega_r)$ for every $r > 0$ and for which

$$|u|_a^b := \sup_{r>0} r^{a+b} |u|_{a,\Omega_r} < \infty. \tag{2.19}$$

A norm of this form is sometimes called a **Schauder norm**. We need some more notation. The space $\mathrm{Lip}_a^{(b-0)}(\Omega)$ is defined as the set of all functions u such that

$$\lim_{r\to 0} r^{a+b} |u|_{a,\Omega_r} = 0.$$

Let Γ be an open cone with vertex at the origin and vertex angle ζ, $0 < \zeta < \pi$. Then $\hat{b}(\zeta,\mu)$ is the largest number $b > 0$ for which there exists a positive function $w(R,\theta) = R^b F(\theta)$ in Γ such that $Lw \le 0$ for all $L \in L_\mu$, where L_μ is the class of real elliptic operators

$$L = \sum_{|\beta|=2} p_\beta(x) D^\beta$$

for which the eigenvalues of the coefficient matrix lie in $[\mu, 1]$, $0 < \mu \le 1$. Theorem 7.2 and 6.1 combined with Remark 2, Section 7 in [GH] provide us with a fundamental result concerning regularity of elliptic problems with respect to Schauder norms. It turns out that if the right–hand side is in some appropriate space $\mathrm{Lip}_a^{(b)}$, then we obtain a jump of two for the solution u in the same scale of Schauder norms, provided that the domain satisfies some cone condition.

Theorem 2.1 *Let $\Omega \subset \mathbf{R}^d$ be a Lipschitz domain satisfying a uniform exterior cone condition with a cone of vertex angle η. Let*

$$L = \sum_{|\beta| \le 2} p_\beta(x) D^\beta$$

be a second order elliptic operator in Ω with principal part contained in L_μ and with coefficients satisfying

$$\begin{aligned} p_\beta &\in Lip_{a-2}^{(0)}(\Omega) \quad \text{if } |\beta| = 2, \\ p_\beta &\in Lip_{a-2}^{(1-0)}(\Omega) \quad \text{if } |\beta| = 1, \\ p_0 &\in Lip_{a-2}^{(2-b)}(\Omega), \end{aligned}$$

where a is a non-integer > 2 and $0 < b < \hat{b}(\pi - \eta, \mu) \le 1$. Furthermore, let $f \in Lip_{a-2}^{2-b} \cap L_2(\Omega)$. Then there exists a constant c such that the solution of

$$\begin{aligned} Lu &= f \quad \text{on} \quad \Omega \subset \mathbf{R}^d, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{2.20}$$

satisfies

$$|u|_a^{(-b)} \le c|f|_{a-2}^{(2-b)}. \tag{2.21}$$

□

Theorem 2.1 can now be used to estimate wavelet coefficients and to characterize the Besov regularity of the solution u of (2.20) for a whole family of operators satisfying the conditions from above. We obtain the following two-dimensional result which essentially means that the spaces $\mathrm{Lip}_a^{(-b)}(\Omega)$ are contained in an appropriate scale of Besov spaces.

Theorem 2.2 *Suppose that the conditions of Theorem 2.1 are satisfied for a domain $\Omega \subset \mathbf{R}^2$ and a second order elliptic operator L on Ω. Then the following holds.*

a) If $a - b \le \frac{3}{2}$, then the solution u of (2.20) satisfies

$$u \in B_\tau^\alpha(L_\tau(\Omega)), \quad \tau = \left(\frac{\alpha}{2} + \frac{1}{2}\right)^{-1}, \quad 0 \le \alpha < 2.$$

b) If $a - b > \frac{3}{2}$, then

$$u \in B_\tau^\alpha(L_\tau(\Omega)), \quad \tau = \left(\frac{\alpha}{2} + \frac{1}{2}\right)^{-1}, \quad 0 \le \alpha < 2(b - a) + 5.$$

Proof: We want to use the characterization (1.16) of Besov spaces. Therefore we have to estimate the coefficients of u with respect to some sufficiently smooth orthonormal wavelet basis. To this end, we first extend u to a function $\tilde{u} \in W^1(L_2(\mathbf{R}^2))$ which is possible since $u \in W^1(L_2(\Omega))$ and the domain is minimally smooth. Since ϕ and ψ are compactly supported, there exists a cube $Q \subset \mathbf{R}^2$, centered at the origin, such that supp $\eta \in Q$ for all $\eta \in \Psi$. Then supp ψ_I is contained in $Q(I) := 2^{-j}k + 2^{-j}[0,1]^2$. Let Λ be the set of pairs (I, η), $I \in \mathcal{D}^+, \eta \in \Psi$ for which $Q(I) \cap \Omega \neq \emptyset$. Then, on $\Omega, \tilde{u}$ possesses a decomposition

$$\tilde{u} = P_0(\tilde{u}) + \sum_{(I,\eta)\in\Lambda} \langle \tilde{u}, \eta_I \rangle \eta_I. \tag{2.22}$$

Let us henceforth assume that ϕ and ψ are at least C^3. Clearly, $P_0(\tilde{u}) \in B^\alpha_\tau(L_\tau(\Omega)), \tau = (\frac{\alpha}{2} + \frac{1}{2})^{-1}$, $0 \leq \alpha \leq 2$, so that it remains to show that

$$\sum_{(I,\eta)\in\Lambda} |\langle \hat{u}, \eta_I \rangle|^\tau < \infty. \tag{2.23}$$

This can be performed by using the ideas developed in [DD]. For any polynomial P_I on $Q(I)$ of total degree < 3 we obtain

$$|\langle \tilde{u}, \eta_I \rangle| = |\langle \tilde{u} - P_I, \eta_I \rangle| \leq \|\tilde{u} - P_I\|_{L_2(Q(I))} \|\eta_I\|_{L_2(Q(I))} = \|u - P_I\|_{L_2(Q(I))}.$$

In general, the best approximation $E_r(F, Q)_p$ of a function $F \in L_p$ by polynomials of order r on a cube Q can be estimated as

$$E_r(F, Q)_p \lesssim |Q|^\mu |F|_{W^r(L_\gamma)}, \qquad \mu := \frac{r}{d} - \frac{1}{\gamma} + \frac{1}{p}. \tag{2.24}$$

We will see below that the intermediate Schauder estimates provide us with some suitable L_∞-estimates. Therefore we will use (2.24) for the case $r = 2,\ p = 2,\ \gamma = \infty,\ d = 2$, i.e., $\mu = 3/2$. Since

$$\text{measure}(\text{supp } \eta_I) \sim |Q(I)| = 2^{-2j},$$

we obtain

$$|\langle \tilde{u}, \eta_I \rangle| \lesssim E_2(\tilde{u}, Q(I))_2 \lesssim |Q(I)|^{3/2} |\tilde{u}|_{W^2(L_\infty)} \lesssim 2^{-3j} |\tilde{u}|_{W^2(L_\infty)}. \tag{2.25}$$

For each refinement level j we cover the interior of Ω by layers of size of order 2^j. More precisely, let Λ_j denote the set of all pairs $(I, \eta) \in \Lambda$ with $|I| = 2^{-2j}$. Furthermore, let $\Lambda_{j,\ell} \subset \Lambda_j$ be the set of those $(I, \eta) \in \Lambda_j$ such that

$$\ell 2^{-j} \leq \delta_I \leq (\ell + 1) 2^{-j} \qquad \text{where} \quad \delta_I := \inf_{x\in Q(I)} \delta(x),$$

and let $\Lambda^0_j := \Lambda_j \backslash \Lambda_{j,0}$. For $(I, \eta) \in \Lambda^0_j$, the corresponding wavelet is supported strictly inside the domain Ω and we may use the Schauder norms to estimate the associated coefficient. Observe that Eq. (2.21) implies that

$$|u|^{(-b)}_a = \sup_{r>0} r^{a-b} |u|_{a,\Omega_r} < \infty,$$

therefore

$$|u|_{a,\Omega_r} \lesssim r^{b-a}.$$

Since $a > 2$, we obtain

$$|u|_{W^\infty(L_2(\Omega_r))} \lesssim \delta^{b-a}. \tag{2.26}$$

Hence if we combine (2.26) and (2.25), we get

$$|\langle \tilde{u}, \eta_I \rangle| \lesssim 2^{-3j} \delta_I^{(b-a)}.$$

Since Ω is a Lipschitz domain, it follows that

$$|\Lambda_{j,\ell}| \lesssim 2^j,$$

therefore

$$\begin{aligned}
\sum_{(I,\eta)\in\Lambda_j^0} |\langle \tilde{u}, \eta_I \rangle|^\tau &\lesssim \sum_{(I,\eta)\in\Lambda_j^0} 2^{-3j\tau} \delta_I^{(b-a)\tau} \\
&\lesssim \sum_{\ell=1}^{\infty} \sum_{(I,\eta)\in\Lambda_{j,\ell}} 2^{-3j\tau} (\ell 2^{-j})^{(b-a)\tau} \\
&\lesssim \sum_{\ell=1}^{\infty} \ell^{(b-a)\tau} 2^j \cdot 2^{-j(3+b-a)\tau} \\
&\lesssim 2^{j(1-(3+b-a)\tau)}.
\end{aligned}$$

Observe that in both of the cases stated in the theorem the conditions on α imply that the series involving ℓ converges. Now we want to sum these expressions over all refinement levels. The series

$$\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_j^0} |\langle \tilde{u}, \eta_I \rangle|^\tau \lesssim \sum_{j=0}^{\infty} 2^{j(1-(3+b-a)\tau)}$$

converges if

$$1 - \tau(3 + b - a) < 0, \quad \text{i.e.,} \quad \tau > \frac{1}{(b-a)+3}.$$

It remains to study the boundary layer. We want to use the following characterization of Sobolev spaces. A functions F is in $W^s(L_2(\mathbf{R}^d))$ if and only if

$$\|P_0(F)\|_{L_2(\mathbf{R}^d)} + \left(\sum_{I\in\mathcal{D}^+} \sum_{\eta\in\Psi} |I|^{-2s/d} |\langle F, \eta_I \rangle|^2 \right)^{1/2} < \infty. \tag{2.27}$$

For the proof of this fact, the reader is e.g. referred to the book of Meyer [M]. Since $\tilde{u} \in W^1(L_2(\mathbf{R}^2))$, we obtain

$$\sum_{I\in\mathcal{D}^+} \sum_{\eta_I\in\Psi} |I|^{-1} |\langle \tilde{u}, \eta_I \rangle|^2 < \infty. \tag{2.28}$$

Using Hölder's inequality with $p = \frac{2}{\tau}$, $q = \frac{2}{2-\tau}$, and the fact that $|\Lambda_{j,0}| \lesssim 2^j$, we get

$$\sum_{(I,\eta)\in\Lambda_{j,0}} |\langle\tilde{u},\eta_I\rangle|^\tau \lesssim 2^{j(\frac{2-\tau}{2})} \left(\sum_{(I,\eta)\in\Lambda_{j,0}} |\langle\tilde{u},\eta_I\rangle|^2 \right)^{\tau/2}.$$

We have to sum these expressions over all refinement levels. If we use Hölder's inequality once again, we find

$$\begin{aligned}\sum_{j=0}^{\infty} \sum_{(I,\eta)\in\Lambda_{j,0}} |\langle\tilde{u},\eta_I\rangle|^\tau &= \sum_{j=0}^{\infty} 2^{\tau j} \left(\sum_{(I,\eta)\in\Lambda_{j,0}} |\langle\tilde{u},\eta_I\rangle|^2 \right)^{\tau/2} 2^{-\tau j} \cdot 2^{j(\frac{2-\tau}{2})} \\ &\lesssim \left(\sum_{j=0}^{\infty} 2^{2j} \sum_{(I,\eta)\in\Lambda_{j,0}} |\langle\tilde{u},\eta_I\rangle|^2 \right)^{\tau/2} \cdot \left(\sum_{j} 2^{(-\frac{2\tau j}{2-\tau}+j)} \right)^{\frac{2-\tau}{2}}.\end{aligned}$$

Since $|I| = 2^{-2j}$ for $(I,\eta) \in \Lambda_{j,0}$ we know from (2.28) that the first sum is finite, and the second one is finite if

$$-\frac{2\tau}{2-\tau} + 1 < 0, \quad \text{i.e.,} \quad \tau > \frac{2}{3}.$$

Since $\frac{1}{(b-a)+3} \leq \frac{2}{3}$ if and only if $(a-b) \leq \frac{3}{2}$, the result follows by employing the relation $\tau = (\frac{\alpha}{2} + \frac{1}{2})^{-1}$. □

Remark 2.1 From the proof of Theorem 2.2 it is clear why this result only holds in two dimensions. The problem is the boundary layer. In the general case it gives rise to the condition $\tau > \frac{2(n-1)}{n+1}$ which is of no use for $n > 2$.

Remark 2.2 Theorem 2.2 essentially states a relation between the regularity of the solution and the vertex angle of the cone. In general, enlarging ζ will produce a smaller value of $\hat{b}(\zeta,\mu)$. Therefore, we loose regularity in the Besov scale if the vertex angle of the exterior cone is too small.

3 Integral Representations and Besov Spaces

In this section, we will estimate the Besov regularity of functions on Lipschitz domains in $\mathbf{R}^2$ that possess an integral representation by a certain kernel. It turns out that the decay of the derivatives of the kernel directly determines the Besov regularity. First of all, we will prove a general theorem that clarifies these relationships. Later on, we will discuss a simple example that illuminates the applicability of these results to the problem (1.1).

Theorem 3.1 *Let Ω be a Lipschitz domain in $\mathbf{R}^2$. Furthermore, let $K : \Omega \times \partial\Omega \to \mathbf{R}$ be a kernel satisfying $K(\,\cdot\,,q) \in C^m(\Omega)$ for some $m \in \mathbf{N}$, and*

$$|D^\beta K(x,q)| \leq \frac{1}{|x-q|^c}, \qquad |\beta| = m, \qquad \textit{for some} \quad c > 1,\ c \in \mathbf{R}. \qquad (3.29)$$

If a function $v(\cdot) \in W^1(L_2(\Omega))$ has a representation

$$v(x) = \int_{\partial\Omega} K(x,q)h(q)dq \tag{3.30}$$

for some function $h(\cdot) \in L_p(\partial\Omega)$, where p, c, and m are related by

$$m + 2 > c + 1/p > (m+3)/2, \tag{3.31}$$

then

$$v(\cdot) \in B^\alpha_\tau(L_\tau(\Omega)), \quad \tau = \left(\frac{\alpha}{2} + \frac{1}{2}\right)^{-1}, \quad 0 < \alpha < \max\{2,\ 2m+3-2c-2/p\}.$$

Proof: Once again, we want to use the characterization (1.16). We will proceed analogously to the proof of Theorem 2.2. Let $\Lambda, \Lambda_j, \Lambda_{j,\ell}$ and Λ_j^0 be defined as above. Since $v \in W^1(L_2(\Omega))$, the boundary layer can be estimated in the same way as in the proof of Theorem 2.2. It remains to study the sets Λ_j^0, $j = 1, 2 \ldots$. Let us assume that the wavelet basis is at least in C^m. It is sufficient to show that

$$|D^\beta v(x)| \leq \delta(x)^{-c+1-1/p}, \qquad |\beta| = m. \tag{3.32}$$

This can be seen as follows. Once again, for any polynomial of total degree $< m$ we obtain

$$|\langle v, \eta_I\rangle| = |\langle v - P_I, \eta_I\rangle| \leq \|v - P_I\|_{L_2(Q(I))}\|\eta_I\|_{L_2(Q(I))}.$$

We want to use expression (2.24) for the case $r = m$, $p = 2$, $d = 2$, $\gamma = \infty$, i.e., $\mu = \frac{m+1}{2}$. By invoking (3.32) we obtain

$$|\langle v, \eta_I\rangle| \lesssim |Q(I)|^{(m+1)/2}|v|_{W^m(L_\infty)} \lesssim 2^{-j(m+1)}\delta_I^{-c+1-1/p}.$$

Therefore by following the lines of the proof of Theorem 2.2 we find

$$\begin{aligned}
\sum_{j=1}^{\infty} \sum_{(I,\eta)\in\Lambda_j^0} |\langle v, \eta_I\rangle|^\tau &\lesssim \sum_{j=0}^{\infty}\sum_{\ell=1}^{\infty} \sum_{(I,\eta)\in\Lambda_{j,\ell}} 2^{-j(m+1)\tau}\delta_I^{(-c+1-1/p)\tau} \\
&\lesssim \sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty} 2^j \cdot 2^{-j(m+1)\tau}(\ell \cdot 2^{-j})^{(-c+1-1/p)\tau} \\
&\lesssim \sum_{j=1}^{\infty} 2^{j(1-(m+2-c-1/p)\tau)} \sum_{\ell=1}^{\infty} \ell^{(-c+1-1/p)\tau} \\
&\lesssim \sum_{j=1}^{\infty} 2^{j(1-(m+2-c-1/p)\tau)} .
\end{aligned}$$

Observe that again Eq. (3.31) and the conditions on α imply that the series involving ℓ converges. The remaining sum is finite if and only if

$$1 - (m+2-c-1/p)\tau < 0, \qquad \text{i.e.,} \quad \tau > \frac{1}{m+2-c-1/p}.$$

To establish (3.32) we first apply Hölder's inequality to (3.30) and obtain by (3.29)

$$\begin{aligned}
|D^\beta v(x)| &\le \int_{\partial\Omega} |D^\beta K(x,q)||h(q)|dq \lesssim \int_{\partial\Omega} \frac{1}{|x-q|^c}|h(q)|dq \\
&\lesssim \left(\int_{\partial\Omega} |h(q)|^p dq\right)^{\frac{1}{p}} \cdot \left(\int_{\partial\Omega} \frac{1}{|x-q|^{\frac{pc}{p-1}}} dq\right)^{\frac{p-1}{p}} \\
&\lesssim \left(\sum_{\ell=0}^{\infty} \int_{B_\ell} \frac{1}{|x-q|^{\frac{pc}{p-1}}} dq\right)^{\frac{p-1}{p}},
\end{aligned}$$

where the sets B_ℓ are defined by

$$B_\ell(x) := \{q \in \partial\Omega \mid \delta(x)2^\ell \le |x-q| \le 2^{\ell+1}\delta(x)\}.$$

On each B_ℓ, we clearly have

$$\frac{1}{|x-q|} \le \delta(x)^{-1}2^{-\ell},$$

and since

$$\text{measure}(B_\ell(x)) \sim 2^\ell \delta(x),$$

we obtain

$$\begin{aligned}
|D^\beta v(x)| &\lesssim \left(\sum_{\ell=0}^{\infty} \text{measure}(B_\ell(x))(\delta(x)2^\ell)^{-\frac{cp}{p-1}}\right)^{\frac{p-1}{p}} \\
&\lesssim \left(\delta(x)^{1-\frac{cp}{p-1}} \sum_{\ell=0}^{\infty} 2^{(1-\frac{cp}{p-1})\ell}\right)^{\frac{p-1}{p}} \\
&\lesssim \delta(x)^{-c+1-\frac{1}{p}},
\end{aligned}$$

and (3.32) is shown. □

To demonstrate how Theorem 3.1 can be used to determine the regularity of the solution u of (1.1) let us discuss the following

Example 3.1 We will consider the boundary value problem (1.17) for the Laplace operator,

$$\begin{aligned}
-\Delta u &= f \quad \text{on} \quad \Omega \subset \mathbf{R}^2, \quad f \in L_2(\Omega), \\
u &= 0 \quad \text{on} \quad \partial\Omega.
\end{aligned} \tag{3.33}$$

First step is the reduction to a Dirichlet problem. We extend f to be zero outside Ω and let

$$\tilde{u} := f * G, \tag{3.34}$$

where

$$G(x) := \frac{1}{2\pi} \log |x| \tag{3.35}$$

is the fundamental solution of the Laplace operator,

$$\Delta G = \delta. \tag{3.36}$$

Then $\tilde{u} \in W^2(L_2(\Omega))$. Let v denote the solution of the Dirichlet problem

$$\begin{aligned} \Delta v &= 0 \quad \text{on} \quad \Omega, \\ v|_{\partial\Omega} &= \tilde{u}|_{\partial\Omega} =: g. \end{aligned} \tag{3.37}$$

In terms of v and $\tilde{u}$, the solution u possesses a decomposition

$$u = \tilde{u} - v. \tag{3.38}$$

We do not have to worry about the "good" part $\tilde{u}$ since $W^2(L_2(\Omega)) = B_2^2(L_2(\Omega)) \hookrightarrow B_2^2(L_\tau(\Omega)) \hookrightarrow B_\tau^\alpha(L_\tau(\Omega))$, $\alpha < 2$. Therefore it remains to study the regularity of the "bad" part v. It can be shown that $\mathrm{Tr}(\nabla\tilde{u})$ belongs to $L_2(\partial\Omega)$, hence $g = \mathrm{Tr}(\tilde{u})$ belongs to $W^1(L_2(\partial\Omega))$. This fact implies that a kernel representation of v satisfying the conditions of Theorem 3.1 exists. This can be seen as follows. For some function h on $\partial\Omega$,

$$\mathcal{S}h(x) := \int_{\partial\Omega} G(x-q)h(q)dq$$

is a harmonic function on Ω. The function $\mathcal{S}h(x)$ is known as the **single layer potential** of h. The properties of $\mathcal{S}$ are summarized in the following theorem which was proved by G. Verchota [V], see also Dahlberg and Kenig [DK].

Theorem 3.2 *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^2$. Furthermore, let f_0 be the unique function in $L_2(\partial\Omega)$ satisfying*

$$(\frac{1}{2}I - \mathcal{K}^*)f_0 = 0 \quad \textit{and} \quad \int_{\partial\Omega} f_0(q)dq = 1,$$

where $\mathcal{K}^*$ is defined by

$$(\mathcal{K}^*f)(p) = p.v. \frac{1}{4\pi}\int_{\partial\Omega} \frac{\langle p-q, N(p)\rangle}{|q-p|^2} f(q)dq,$$

and $N(p)$ denotes the outward normal. If $\mathcal{S}(f_0) \neq 0$ in Ω, then there exists an $\epsilon > 0$ which depends only on the Lipschitz character of Ω, such that

$$\mathcal{S}: \ L_p(\partial\Omega) \longrightarrow W^1(L_p(\partial\Omega)), \qquad 1 < p \leq 2+\epsilon$$

is an invertible operator.

It follows from Theorem 3.2 that at least for domains satisfying $\mathcal{S}(f_0) \neq 0$ the solution v of (3.37) is given by

$$v(x) = \int_{\partial\Omega} G(x-q)\mathcal{S}^{-1}(g)(q)dq.$$

We set

$$K(x,q) := G(x-q), \qquad h(q) := \mathcal{S}^{-1}(g)(q).$$

It is straightforward to check that for instance

$$|D^\beta G(x-q)| \le \frac{1}{|x-q|^3}, \quad |\beta| = 3.$$

Then the conditions of Theorem 3.1 are satisfied with $c = m = 3$ and $p = 2$, i.e., $2m + 3 - 2c - 2/p = 2$. Consequently,

$$u \in B^\alpha_\tau(L_\tau(\Omega)), \quad \tau = \left(\frac{\alpha}{2} + \frac{1}{2}\right)^{-1}, \quad 0 \le \alpha < 2.$$

Using a quite different approach, a similar result has also been established in [DD].

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