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Multiscale Characterizations of Besov Spaces on Bounded Domains

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We characterize the Besov regularity of functions on Lipschitz domains by means of their error of approximation by certain sequences of operators. As an application, we consider wavelet decompositions and we characterize Besov quasi-norms in terms of weighted sequence norms.

1. INTRODUCTION

The theory of wavelets and multiresolution analysis is usually developed on $\mathbb{R}^d$. However, applications of wavelets to image processing and numerical methods for partial differential equations require multiresolution analysis on domains or manifolds in $\mathbb{R}^d$. The study of multiresolution in these settings is just beginning. Building on the construction of multiresolution on intervals (and cubes in $\mathbb{R}^d$), Cohen *et al.* [2] have constructed a multiresolution which applies to a fairly large class of domains Ω (basically coordinatewise Lipschitz) in $\mathbb{R}^d$. They have shown in their analysis that various smoothness spaces can be characterized by this multiresolution. For example, their analysis applies to the Besov spaces $B^\alpha_q((L_p)(\Omega)$ provided $p \geqslant 1$. However, the same Besov spaces with $p<1$ are also important in analysis, especially in analyzing nonlinear methods [5] such as image compression [4] or noise removal [10].

The purpose of the present paper is to show that the Besov spaces for $p<1$ can also be characterized in the usual way by multiresolution analysis. To prove this, we analyze the approximation properties of certain sequences of operators T_j, $j \in \mathbb{N}$, which include as a special case the projectors in the Cohen *et al.* multiresolution. The operators take the form

$$T_j f := \sum_{\gamma \in G_j} \langle f, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d} \, \Phi_\gamma, \qquad j \in \mathbb{N}, \tag{1.1}$$

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where Φ_γ, $\tilde{\Phi}_\gamma$, $\gamma \in G_j$, are compactly supported functions and G_j is a subset of $\mathscr{L}_j := 2^{-j}\mathbb{Z}^d$, $j \in \mathbb{Z}^d$.

A special case of our results is the characterization of the Besov spaces $B^\alpha_q(L_p(\Omega))$ for $0 < p \leqslant 1$, $0 < q \leqslant \infty$, and $\alpha > d(1/p - 1)$, on the class of domains considered in [2]. The restrictions on the parameters in the Besov space imply that this Besov space is embedded in L_1. For $\Omega = \mathbb{R}^d$ this kind of problem has been already addressed in various settings in [1, 6, 11, 12, 13].

Throughout this paper, we shall use standard multivariate notation; for every $x = (x_1, ..., x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, we define $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and $D^\alpha := \partial^{|\alpha|}/(\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d)$.

Let Ω be a bounded, open subset of $\mathbb{R}^d$. As usual for any function $f \in L_p(\Omega)$ and $r \in \mathbb{N}$ we denote by $\omega_r(f, t)_p$, $t > 0$, its rth modulus of smoothness

$$\omega_r(f, t)_p := \omega_r(f, t, \Omega)_p := \sup_{|h| \leqslant t} \|\Delta^r_h(f, \cdot, \Omega)\|_{L_p(\Omega)}, \tag{1.2}$$

where $\Delta^r_h(f, \cdot, \Omega)$ is the rth forward difference relative to Ω which is defined by

$$\Delta^r_h(f, x, \Omega) := \begin{cases} \Delta^r_h(f, x), & x, x+h, ..., x+rh \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Let now $s > 0$, $0 < p$, $q \leqslant \infty$, and assume that r is an integer greater than s. The Besov space $B^s_q(L_p(\Omega))$ is defined to be the collection of all functions f in $L_p(\Omega)$ such that the (quasi-) norm

$$\|f\|_{B^s_q(L_p(\Omega))} := \|f\|_{L_p(\Omega)} + \left(\sum_{k=1}^\infty [2^{ks}\omega_r(f, 2^{-k}, \Omega)_p]^q \right)^{1/q} \tag{1.3}$$

is finite (where the *summation* is replaced by a *supremum* when $q = \infty$).

It is easily seen that the sum in (1.3) defines a semi- (quasi-) norm which we denote by $|\cdot|_{B^s_q(L_p(\Omega))}$; we further define

$$\|f\|_{B^s_q(L_p(\Omega))} =: \|f\|_{L_p(\Omega)} + |f|_{B^s_q(L_p(\Omega))}. \tag{1.4}$$

It is also known that different values of $r > s$ yield equivalent norms.

Finally by $A \approx B$ we mean that there exist positive constants, independent of the variables involved, such that const $\leqslant A/B \leqslant$ const.

2. CHARACTERIZATION OF BESOV SPACES BY MEANS OF T_j, $j \in \mathbb{N}$

In the present section we intend to establish, under certain assumptions on the operators T_j, $j \in \mathbb{N}$, that for every $0 < p \leqslant 1$, $0 < q \leqslant \infty$, $\alpha > d(1/p - 1)$, the following three quantities are equivalent quasi-norms on $B_q^\alpha(L_p(\Omega))$:

$$\text{(2.1)} \qquad \begin{aligned} &\text{(i)} \quad \|f\|^q_{B_q^\alpha(L_p(\Omega))}, \\ &\text{(ii)} \quad \sum_{j \geqslant -1} 2^{j\alpha q} \, \|f - T_j f\|^q_{L_p(\Omega)}, \\ &\text{(iii)} \quad \sum_{j \geqslant 0} 2^{j\alpha q} \, \|T_j f - T_{j-1} f\|^q_{L_p(\Omega)}, \end{aligned}$$

with constants of equivalency possibly depending on α, p, q, and d. We use here and later the convention that $T_{-1} := 0$.

We are going to impose two types of assumptions on the operators T_j; the first is pertinent to the establishment of direct theorems, referred to as *Jackson inequalities*, while the second type is suitable for the proof of inverse theorems, usually associated with the name of *Bernstein*, who established analogous results for polynomials.

Direct Theorems

We begin by considering first the case when $\Omega = \mathbb{R}^d$. We assume that we have in hand a sequence of linear operators L_j, $j \in \mathbb{N}$, given by

$$L_j f := \sum_{\gamma \in \Gamma_j} \langle f, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d} \, \Phi_\gamma, \tag{2.2}$$

where $\Gamma_j \subset 2^{-j} \mathbb{Z}^d$ and for each $\gamma \in \Gamma_j$, the functions Φ_γ, $\tilde{\Phi}_\gamma$ satisfy the following two properties:

(J1) There exist absolute constants such that:

$$|\mathrm{diam}(\mathrm{supp}(\Phi_\gamma))| \leqslant \text{const } 2^{-j}, \qquad |\mathrm{diam}(\mathrm{supp}(\tilde{\Phi}_{\gamma)}))| \leqslant \text{const } 2^{-j}.$$

(J2) There exists an absolute constant such that for each $\mu \in (0, \infty]$,

$$\|\Phi_\gamma\|_{L_\mu(\mathbb{R}^d)} \leqslant \text{const } 2^{jd(1/2 - 1/\mu)}, \qquad \|\tilde{\Phi}_\gamma\|_{L_\mu(\mathbb{R}^d)} \leqslant \text{const } 2^{jd(1/2 - 1/\mu)}.$$

Note. The case of general μ follows from the case $\mu = \infty$ in this condition while (J1)–(J2) together guarantee that L_j, $j \in \mathbb{N}$, are defined on $L_1(\mathrm{loc})$.

The reader should keep in mind the biorthogonal wavelets as typical candidates for the functions Φ_γ and $\tilde{\Phi}_\gamma$; however, there are several other important examples in which we shall also be interested. We also need to make a further assumption regarding the approximation properties of the sequence L_j, $j \in \mathbb{N}$:

(J3) There is an integer $N \geqslant 0$ such that for each $j \in \mathbb{N}$, and every polynomial $P \in \Pi_N$, we have

$$L_j(P, x) = P(x), \qquad x \in \mathbb{R}^d. \tag{2.3}$$

Note that the larger the integer N is then the larger the polynomial reproduction and the better the approximation properties of the L_j, $j \in \mathbb{N}$.

We are ready now to state our first theorem:

THEOREM 2.4. *Let $0 < p \leqslant 1$, $\alpha > 0$, and $N \in \mathbb{N}$ be such that $N + 1 > \alpha \geqslant d(1/p - 1)$. If L_j, $j \in \mathbb{N}$, are defined by* (2.2) *and satisfy conditions* (J1)–(J3) *above, then for every $f \in B_p^\alpha(L_p(\mathbb{R}^d))$*

$$\|f - L_j(f)\|_{L_p(\mathbb{R}^d)} \leqslant \text{const}\ 2^{-j\alpha}\ |f|_{B_p^\alpha(L_p(\mathbb{R}^d))}, \qquad j = 1, 2, ..., \tag{2.5}$$

for some constant independent of f and j.

Although Theorem 2.4 is important by itself, here we will use it primarily as a vehicle for establishing Jackson's inequality for sequences of operators defined on domains in $\mathbb{R}^d$. We shall limit our development to minimally smooth domains in the sense of Stein (see [14] for their definition and properties). In this setting, we let T_j, $j \in \mathbb{N}$, be a sequence of linear operators of the form (1.1). We shall assume that:

(J4) For each $\gamma \in G_j$, supp $\tilde{\Phi}_\gamma \subset \Omega$.

(J5) For each $j \in \mathbb{N}$, there exists an extension Γ_j of the set G_j ($G_j \subset \Gamma_j \subset 2^{-j} \mathbb{Z}^d$) such that the (extension) operator

$$E_j f := \sum_{\gamma \in \Gamma_j} \langle f, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d} \Phi_\gamma \tag{2.6}$$

satisfies

$$E_j(f)(x) = T_j(f)(x), \qquad x \in \Omega.$$

THEOREM 2.7. *Let $(T_j)_{j \in \mathbb{N}}$ be a sequence of linear operators of the form* (1.1) *and let Ω be a minimally smooth domain. Let also $0 < p \leqslant 1$, $\alpha > 0$, and $N \in \mathbb{N}$ be such that $N + 1 > \alpha \geqslant d(1/p - 1)$. If $(T_j)_{j \in \mathbb{N}}$ satisfy assumptions* (J4)–(J5) *and the extension operators E_j, $j \in \mathbb{N}$, satisfy* (J1)–(J3) *then, for every $f \in B_p^\alpha(L_p(\Omega))$*

$$\|f - T_j(f)\|_{L_p(\Omega)} \leqslant \text{const}\ 2^{-j\alpha}\ \|f\|_{B_p^\alpha(L_p(\Omega))}, \qquad j = 1, 2, ..., \tag{2.8}$$

for some constant independent of f and j.

We start by establishing first the proof of Theorem 2.4. For this we will take advantage of the local approximation properties of polynomials. We denote by $\mathcal{D}_j$ the set of dyadic cubes in $\mathbb{R}^d$ with sidelength 2^{-j}. If $Q \in \mathcal{D}_j$, we denote by Λ_Q the set of all indices $\gamma \in 2^{-j}\mathbb{Z}^d$ such that the support of Φ_γ intersects Q. We note for further use that

$$\# \Lambda_Q \leqslant \text{const}. \tag{2.9}$$

We also let $\tilde{Q}$ be the smallest cube containing $(\bigcup_{\gamma \in \Lambda_Q} \operatorname{supp} \tilde{\Phi}_\gamma) \cup Q$. Then, (J1) guarantees that $|\tilde{Q}| \leqslant \text{const}\, 2^{-jd}$.

For $\beta := d(1/p - 1)$ and $N+1 > \beta$, it is well known (see [8]) that any polynomial $P \in \Pi_N$ of best $L_p(\tilde{Q})$ approximation to f satisfies the local estimate

$$\|f - P\|_{L_1(\tilde{Q})} \leqslant \text{const}\, |f|_{B_p^\beta(L_p(\tilde{Q}))}. \tag{2.10}$$

Using (2.10) we will prove the following lemma:

LEMMA 2.11. *Let* $0 < p \leqslant 1$ *and* $\beta := d(1/p - 1)$. *There exists a constant such that for every* $f \in B_p^\beta(L_p(\tilde{Q}))$, $Q \in D_j$,

$$\|f - L_j(f)\|_{L_p(Q)} \leqslant \text{const}\, 2^{-j\beta}\, |f|_{B_p^\beta(L_p(\tilde{Q}))}, \qquad j = 1, 2, \ldots. \tag{2.12}$$

Proof. Let P be a polynomial in Π_N satisfying (2.10). We have

$$\|f - L_j(f)\|_{L_p(Q)}^p \leqslant \|f - P\|_{L_p(Q)}^p + \|L_j(f) - P\|_{L_p(Q)}^p. \tag{2.13}$$

From Hölder's inequality and (2.9), we have that

$$\begin{aligned} \|f - P\|_{L_p(Q)}^p &\leqslant \left(\int_{\tilde{Q}} |f - P| \right)^p |\tilde{Q}|^{1-p} \\ &\leqslant \text{const}\, 2^{-jd(1-p)}\, |f|_{B_p^\beta(L_p(\tilde{Q}))}^p. \end{aligned} \tag{2.14}$$

On the other hand, employing the polynomial-reproduction of L_j, it follows that

$$\|L_j(f) - P\|_{L_p(Q)}^p = \|L_j(f - P)\|_{L_p(Q)}^p.$$

Moreover, on Q we have with the help of (J2) that

$$\begin{aligned} |L_j(f-P)(\cdot)|^p &\leqslant \sum_{\gamma \in \Lambda_Q} \left(\int_{\tilde{Q}} |f - P|\, |\tilde{\Phi}_\gamma| \right)^p |\Phi_\gamma(\cdot)|^p \\ &\leqslant \text{const}\, 2^{jdp/2} \left(\int_{\tilde{Q}} |f - P| \right)^p \sum_{\gamma \in \Lambda_Q} |\Phi_\gamma(\cdot)|^p. \end{aligned}$$

Integrating the last inequality, and taking into account (2.10) and (J2), we get that

$$\|L_j(f-P)\|_{L_p(Q)}^p \leqslant \text{const}\, 2^{-jd(1-p)} \left(\int_{\tilde{Q}} |f-P|\right)^p \leqslant \text{const}\, 2^{-jd(1-p)}\, |f|_{B_p^\beta(L_p(\tilde{Q}))}^p. \tag{2.15}$$

Employing (2.14) and (2.15) in (2.13) the result follows. ∎

LEMMA 2.16. *Let* $\alpha > \beta := d(1/p-1)$. *For each cube* Q *and* $N > \alpha - 1$, *there exists a polynomial* $P := P_Q$ *of degree* $\leqslant N$ *such that*

$$|f-P|_{B_p^\beta(L_p(Q))} \leqslant \text{const}\, |Q|^{(\alpha-\beta)/d}\, |f|_{B_p^\alpha(L_p(Q))}. \tag{2.17}$$

Proof. Without loss of generality we assume that $Q=[0,1]^d$; otherwise the result follows by dilation and translation.

Let $S_k(f)$, $k \in \mathbb{N}$, be a best approximation to f from the space of smooth dyadic splines of coordinate degree N on the partition $2^{-k}\mathbb{Z}^d \cap [0,1]^d$. It is well known (see [8]) that for every $\gamma < N+1$ and $0 < q \leqslant \infty$ there exist constants such that

$$|f|_{B_q^\gamma(L_p([0,1]^d))}^q \approx \sum_{k \geqslant 0} 2^{k\gamma q}\, \|S_{k+1}(f) - S_k(f)\|_{L_p([0,1]^d)}^q.$$

Moreover, the proof of Theorem 4.8 in [8] shows that

$$\begin{aligned}
|f - S_0(f)|_{B_q^\beta(L_p([0,1]^d))}^q &\leqslant \text{const} \sum_{k \geqslant 0} 2^{k\beta q}\, \|S_{k+1}(f) - S_k(f)\|_{L_p([0,1]^d)}^q \\
&\leqslant \text{const} \sum_{k \geqslant 0} 2^{k\alpha q}\, \|S_{k+1}(f) - S_k(f)\|_{L_p([0,1]^d)}^q \\
&\leqslant \text{const}\, |f|_{B_q^\alpha(L_p([0,1]^d))}^q.
\end{aligned}$$

Since $S_0(f)$ is just a polynomial of coordinate degree $\leqslant N$ on $[0,1]^d$, we arrive at (2.17) for $Q=[0,1]^d$. ∎

We are ready now to prove the two theorems of this section:

Proof of Theorem 2.4. We let $Q \in D_j$ and we assume that $P \in \Pi_N$ satisfies (2.17) with respect to Q. Since $L_j(P) = P$, (2.12) combined with (2.17) gives that

$$\begin{aligned}
\|f - L_j(f)\|_{L_p(Q)}^p &= \|f - P - L_j(f-P)\|_{L_p(Q)}^p \\
&\leqslant \text{const}\, 2^{-j\beta p}\, |f-P|_{B_p^\beta(L_p(\tilde{Q}))}^p \\
&\leqslant \text{const}\, 2^{-j\alpha p}\, |f|_{B_p^\alpha(L_p(\tilde{Q}))}^p.
\end{aligned} \tag{2.18}$$

Noting that any point $x \in \mathbb{R}^d$ lies in at most a constant number (independent of j) of cubes $\tilde{Q}$ we can add the estimates in (2.18) over all disjoint Q's and get the desired result from the subadditivity of $|f|^p_{B^\alpha_p(L_p(\tilde{Q}))}$.

Proof of Theorem 2.7. Let $f \in B^\alpha_p(L_p(\Omega))$. The smoothness of the boundary of Ω guarantees (see [9]) the existence of an extension $\tilde{f}$ of f to $\mathbb{R}^d$ with $\tilde{f}_{|\Omega} = f$ and

$$\|\tilde{f}\|_{B^\alpha_p(L_p(\mathbb{R}^d))} \leqslant \text{const } \|f\|_{B^\alpha_p(L_p(\Omega))}. \tag{2.19}$$

Since for every $j \in \mathbb{N}$, $E_j \tilde{f} = T_j f$ on Ω it follows from (J4) and Theorem 2.4 (with E_j instead of L_j, $j \in \mathbb{N}$) that

$$\begin{aligned} \|f - T_j f\|_{L_p(\Omega)} &= \|\tilde{f} - E_j \tilde{f}\|_{L_p(\Omega)} \\ &\leqslant \|\tilde{f} - E_j \tilde{f}\|_{L_p(\mathbb{R}^d)} \\ &\leqslant \text{const } 2^{-j\alpha} \|\tilde{f}\|_{B^\alpha_p(L_p(\mathbb{R}^d))}. \end{aligned} \tag{2.20}$$

From (2.19) and (2.20) we conclude that

$$\|f - T_j f\|_{L_p(\Omega)} \leqslant \text{const } 2^{-j\alpha} \|f\|_{B^\alpha_p(L_p(\Omega))}. \tag{2.21}$$

Inverse Theorems

Next, we turn our attention to the establishment of Bernstein type inequalities. Let again T_j, $j \in \mathbb{N}$, be the family of linear operators defined on the domain Ω by

$$T_j f := \sum_{\gamma \in G_j} \langle f, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d} \Phi_\gamma, \qquad j \in \mathbb{N}.$$

Our main assumptions on the operators $\{T_j\}$ are the following:

(B1) The family of functions $\{\Phi_\gamma : \gamma \in G_j\}$ is uniformly L_p-stable; i.e., there exist positive constants independent of j such that for every sequence of complex numbers $b := (b(\gamma))_{\gamma \in G_j}$

$$\|b\|_{l_p(G_j)} \approx 2^{jd(1/p - 1/2)} \left\| \sum_{\gamma \in G_j} b(\gamma) \Phi_\gamma \right\|_{L_p(\Omega)}. \tag{2.22}$$

(B2) For each $j \in \mathbb{N}$ there are complex numbers $b(\gamma)$, $\gamma \in G_{j+1}$, such that

$$T_{j+1}(f) - T_j(f) = \sum_{\gamma \in G_{j+1}} b(\gamma) \Phi_\gamma.$$

(B3) For every $\gamma \in G_j$ and $j \in \mathbb{N}$,

(2.23) $$\omega_r(\Phi_\gamma, t)_p \leqslant \text{const}\ 2^{jd(1/2-1/p)} \min\{1, (2^j t)^r\},$$

for some constant independent of j.

These assumptions are natural in the setting of a multiresolution analysis. Condition (B2) is related to the nesting of the spaces $V_j(\Omega) \subset V_{j+1}(\Omega)$, $j \in \mathbb{N}$. Condition (B3) imposes, roughly speaking, smoothness of order r on the functions Φ_γ; for instance if $\Phi_\gamma(\cdot) := 2^{jd/2}\phi(2^j \cdot -k)$, $k \in \mathbb{Z}^d$, with ϕ a compactly supported function in $C^r(\mathbb{R}^d)$, it is easily seen that

(2.24) $$\begin{aligned}\omega_r(\Phi_\gamma, t)_p &\leqslant \text{const}\ 2^{jd(1/2-1/p)}\omega_r(\phi, 2^j t)_p \\ &\leqslant \text{const}\ 2^{jd(1/2-1/p)} \min\{1, (2^j t)^r\}.\end{aligned}$$

THEOREM 2.25. *Let* (T_j), $j \in \mathbb{N}$, *be a sequence of operators defined as in* (1.1). *We assume that the family* Φ_γ, $\gamma \in G_j$, $j \in \mathbb{N}$, *satisfies* (B1–B3) *above. Then for every* $f \in L_p(\Omega)$, $k \in \mathbb{N}$, *and* $0 < \mu \leqslant p \leqslant 1$,

(2.26) $$\omega_r(f, 2^{-k})_p^\mu \leqslant \mathit{const}\ 2^{-kr\mu} \sum_{j=-1}^{k} 2^{jr\mu} \|f - T_j(f)\|_{L_p(\Omega)}^\mu,$$

where $T_{-1} := 0$.

Proof. Using the telescoping summation

$$f = f - T_k(f) + \sum_{j=0}^{k} T_j(f) - T_{j-1}(f),$$

we have

(2.27) $$\Delta_h^r(f, x) = \Delta_h^r(f - T_k(f), x) + \sum_{j=0}^{k} \Delta_h^r(t_j(f), x),$$

where $t_j(f) := T_j(f) - T_{j-1}(f)$. It follows that

(2.28) $$\begin{aligned}\|\Delta_h^r(f, x)\|_p^p(\Omega) &\leqslant \text{const}\ \|f - T_k(f)\|_{L_p(\Omega)}^p \\ &\quad + \sum_{j=0}^{k} \|\Delta_h^r(t_j(f), x)\|_p^p(\Omega).\end{aligned}$$

Let $b(\gamma)$, $\gamma \in G_{j+1}$, be the numbers appearing in the representation (B2). Then,

$$\|\Delta_h^r(t_j(f), x)\|_p^p(\Omega) \leqslant \sum_{\gamma \in G_{j+1}} |b(\gamma)|^p \|\Delta_h^r(\Phi_\gamma, x)\|_p^p(\mathbb{R}^d).$$

Taking, on both sides, a supremum over $|h| \leqslant t$ and using (B1), (B3) we get

$$\begin{aligned}
(2.29) \qquad \omega_r(t_j(f), t)_p^p &\leqslant \sum_{\gamma \in G_{j+1}} |b(\gamma)|^p \, \omega_r(\Phi_\gamma, t)_p^p \\
&\leqslant \text{const} \min\{1, (2^j t)^{rp}\} \, \|t_j(f)\|_{L_p(\Omega)}^p.
\end{aligned}$$

Taking a supremum over $|h| \leqslant 2^{-k}$ in (2.28) and using this in (2.29), we conclude that for every $k \in \mathbb{N}$ and $0 < \mu \leqslant p \leqslant 1$

$$\begin{aligned}
\omega_r(f, 2^{-k})_p &\leqslant \text{const} \left(\|f - T_k(f)\|_{L_p(\Omega)}^p + \sum_{j=0}^{k} \omega_r(t_j(f), 2^{-k})_p^p \right)^{1/p} \\
&\leqslant \text{const} \left(\|f - T_k(f)\|_{L_p(\Omega)}^p + \sum_{j=0}^{k} 2^{jrp} 2^{-krp} \, \|t_j(f)\|_{L_p(\Omega)}^p \right)^{1/p} \\
&\leqslant \text{const} \, 2^{-kr} \left(\sum_{j=-1}^{k} 2^{jrp} \, \|f - T_j(f)\|_{L_p(\Omega)}^p \right)^{1/p} \\
&\leqslant \text{const} \, 2^{-kr} \left(\sum_{j=-1}^{k} 2^{jr\mu} \, \|f - T_j(f)\|_{L_p(\Omega)}^\mu \right)^{1/\mu}. \quad \blacksquare
\end{aligned}$$

Having established Theorems 2.7 and 2.25 we are ready to prove the equivalence of the semi-norms in (2.1). For this we need to assume that family T_j, $j \in \mathbb{N}$, satisfies the assumptions of both theorems.

THEOREM 2.30. *Let Ω be a minimally smooth domain and assume that $0 < p \leqslant 1$, $\alpha > d(1/p - 1)$, and $0 < q \leqslant \infty$. If the family of operators T_j, $j \in \mathbb{N}$, satisfies conditions* (J1)–(J5) *of Theorem* 2.7 *and* (B1)–(B3) *of Theorem* 2.25 *with* $\min\{N+1, r\} > \alpha$, *then the following quasi-norms are equivalent*:

(i) $\|f\|_{B_q^\alpha(L_p(\Omega))}^q$,

(ii) $\sum_{j \geqslant -1} 2^{j\alpha q} \, \|f - T_j f\|_{L_p(\Omega)}^q$,

(iii) $\sum_{j \geqslant 0} 2^{j\alpha q} \, \|T_j f - T_{j-1} f\|_{L_p(\Omega)}^q$,

where the constants of equivalency are independent of $f \in B_q^\alpha(L_p(\Omega))$.

Proof. The equivalence of (i) and (ii) follows from Jackson's (2.12) and Bernstein's (2.26) inequalities. Since the arguments involved are well known we refer the reader to [7] for details. As far as the equivalence of (ii) and (iii) is concerned, one direction follows from $\|T_j f - T_{j-1} f\|_{L_p} \leqslant \text{const}\{\|T_j f - f\|_{L_p} + \|T_{j-1} f - f\|_{L_p}\}$ and the other from the so-called Hardy's inequality (see [8] for details). $\blacksquare$

3. WAVELET-DECOMPOSITIONS ON SMOOTH DOMAINS

We assume that the reader is familiar with the theory of orthogonal wavelets and briefly review the construction of biorthogonal wavelet-bases on bounded domains of $\mathbb{R}^d$ as it was proposed in [2]. As is customary, in what follows for any function f defined on $\mathbb{R}^d$ we use the notation $f_{j,k}(\cdot) := 2^{jd/2} f(2^j \cdot - k)$.

The term *biorthogonal wavelets* refers to a pair of functions $\psi, \tilde{\psi}$ in $L_2(\mathbb{R})$ that satisfy the duality principle

$$\int_{\mathbb{R}} \psi(x) \overline{\tilde{\psi}(x-k)}\, dx = \delta_{0,k}, \qquad k \in \mathbb{Z}, \tag{3.1}$$

and whose dilated translates $\{\psi_{j,k}\}_{(j,k) \in \mathbb{Z} \times \mathbb{Z}}$ constitute a Riesz basis for $L_2(\mathbb{R})$. Then, $\{\tilde{\psi}_{j,k}\}_{(j,k) \in \mathbb{Z} \times \mathbb{Z}}$ is the dual basis.

In other words every function $f \in L_2(\mathbb{R})$ enjoys the unique decomposition

$$f = \sum_{j,k \in \mathbb{Z}} c_{j,k}(f)\, \psi_{j,k}, \qquad c_{j,k} := \int_{\mathbb{R}} f(y) \overline{\tilde{\psi}_{j,k}(y)}\, dy, \tag{3.2}$$

and for some constants independent of f

$$\text{const} \sum_{j,k \in \mathbb{Z}} |c_{j,k}|^2 \leqslant \|f\|_{L_2}^2 \leqslant \text{const} \sum_{j,k \in \mathbb{Z}} |c_{j,k}|^2.$$

The original construction of biorthogonal wavelets on $\mathbb{R}$, given in [3], is built on two functions $\varphi, \tilde{\varphi} \in L_2(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} \varphi(x)\, \tilde{\varphi}(x-k)\, dx = \delta_{0,k}, \qquad k \in \mathbb{Z}. \tag{3.3}$$

It is also required that both $\varphi, \tilde{\varphi}$ satisfy the refinement equations

$$\varphi(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \varphi(2x-k), \qquad \tilde{\varphi}(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} \tilde{h}_k \tilde{\varphi}(2x-k), \tag{3.4}$$

(for some sequences of complex numbers (h_k) and $(\tilde{h}_k)$, respectively) and that their integer translates (*shifts*) form stable systems.

The functions $\phi, \tilde{\phi}$ are then used to construct a dual multiresolution system consisting of two ascending sequences of subspaces (V_j), $(\tilde{V}_j)$, $j \in \mathbb{Z}$, of $L_2(\mathbb{R})$. Both sequences are generated from the $2^{-j}\mathbb{Z}$-shifts of the functions $\varphi_{j,0} := 2^{j/2}\varphi(2^j x)$ and $\tilde{\varphi}_{j,0} := 2^{j/2}\tilde{\varphi}(2^j x)$, respectively: thus,

$$V_j \subset V_{j+1}, \qquad \tilde{V}_j \subset \tilde{V}_{j+1}, \qquad j \in \mathbb{Z}, \tag{3.5}$$

and

$$V_j := \overline{\operatorname{span}}\{\varphi_{j,k}\},\ j \in \mathbb{Z}, \qquad \tilde{V}_j := \overline{\operatorname{span}}\{\tilde{\varphi}_{j,k}\},\ j \in \mathbb{Z}.$$

We further assume that $\varphi, \tilde{\varphi}$ are compactly supported functions whose Fourier transform does not vanish at the origin, i.e., $\hat{\phi}(0), \hat{\tilde{\phi}}(0) \neq 0$; these assumptions guarantee that

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \qquad \text{and} \qquad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L_2(\mathbb{R}). \tag{3.6}$$

Starting with such a biorthogonal multiresolution ladder, wavelet analysis constructs a certain (oblique) complementing subspace W_j of V_j in V_{j+1}, $V_{j+1} = V_j + W_j$, with W_j generated by the wavelet function ψ and its dual space $\tilde{W}_j$ generated by $\tilde{\psi}$. One then has the decomposition

$$L_2(\mathbb{R}) = \sum_{j \in \mathbb{Z}} W_j,$$

from which one derives (3.2).

Turning our attention to the multivariate case, we can obtain biorthogonal wavelet bases by means of tensor products. For $x := (x_1, ..., x_d)$ we define

$$\phi(x) := \varphi(x_1) \cdots \varphi(x_d), \qquad \tilde{\phi}(x) := \tilde{\varphi}(x_1) \cdots \tilde{\varphi}(x_d),$$

where φ and $\tilde{\varphi}$ are the functions defined above. It is trivially seen that ϕ and $\tilde{\phi}$ satisfy the refinement equations (3.4) with certain coefficients a_k, $\tilde{a}_k$, respectively (for instance $a_k := h_{k_1} \cdots h_{k_d}$, $k := (k_1, ..., k_d)$). Moreover, the sequences

$$V_j := \overline{\operatorname{span}}\{\phi_{j,k}\}, \quad j \in \mathbb{Z}, \qquad \tilde{V}_j := \overline{\operatorname{span}}\{\tilde{\phi}_{j,k}\}, \quad j \in \mathbb{Z},$$

both form multiresolution ladders for $L_2(\mathbb{R}^d)$.

To describe the corresponding tensor-product wavelets we set $\eta_0 := \varphi$, $\eta_1 := \psi$, and we let E be the set of nonzero vertices of the unit cube $[0, 1]^d$. A family of $2^d - 1$ wavelets is then given by

$$\Psi^e(x) := \prod_{j=1}^{d} \eta_{e_j}(x_j), \qquad e = (e_1, ..., e_d) \in E.$$

In a similar manner, one defines the dual functions $\hat{\Psi}^e$, $e \in E$.

The family $\{\Psi^e_{j,k}\}_{e,j,k}$ is a Riesz basis for $L_2(\mathbb{R}^d)$ and in analogy to (3.2) for every $f \in L_2(\mathbb{R}^d)$, we have

$$f = \sum_{e \in E} \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} c_{j,k,e}(f)\, \Psi^e_{j,k}, \qquad c_{j,k,e} := \int_{\mathbb{R}^d} f(y)\, \overline{\tilde{\Psi}^e_{j,k}(y)}\, dy. \tag{3.7}$$

In several applications of multiscale analysis such as in image processing and the numerical solutions of PDEs one needs a corresponding theory relative to bounded domains, that preserves all the important features of wavelets; in particular, when nonlinear methods are used in data compression (see [5]) it is of paramount importance to reflect the smoothness of functions in the Besov spaces $B^r_q(L_p(\Omega))$ in their wavelet coefficients. However, to achieve this we need some additional conditions on the functions ϕ and $\tilde{\phi}$ mostly taken from [2]. We list these properties below and we assume that they hold for the rest of the paper.

(P1) $\phi(0) \neq 0$ and there exists an integer $L > 0$ such that,

$$\operatorname{supp}(\phi), \qquad \operatorname{supp}(\tilde{\phi}) \subset [-L, L]^d.$$

(P2) ϕ satisfies the Strang–Fix conditions of order $N+1$, i.e.,

$$\hat{\phi}(2\pi k) = 0, \qquad k \in \mathbb{Z}^d \backslash \{0\}, \qquad n = 0, ..., N.$$

(P3) Both ϕ and $\tilde{\phi}$ satisfy

$$\hat{\phi}(0) = 1, \qquad \hat{\tilde{\phi}}(0) = 1,$$

while for $1 \leqslant |\beta| < N+1$,

$$D^\beta \hat{\tilde{\phi}}(0) = D^\beta \hat{\phi}(0) = 0.$$

(P4) For some $r > \alpha$, $\phi \in C^r(\mathbb{R}^d)$.

(P5) The shifts (integer translates) of ϕ are locally linearly independent; that is, for any cube $Q \subset \mathbb{R}^d$ the family of functions

$$\{\phi(\cdot - j)\colon j \in \mathbb{Z}^d, \text{ and } \phi(\cdot - j) \text{ is not identically zero on } Q\}$$

is linearly independent over Q.

We are ready now to describe the construction of a multiresolution analysis on bounded domains as it was proposed in [2]. In that paper, they have constructed a ladder of spaces $V_j(\Omega)$, $j = 1, ...$, which retains the important properties of multiresolution.

Let Ω be a fixed bounded domain. For each $j \in \mathbb{N}$, we define the sets

$$\Omega_j := \{2^{-j}k\colon k \in \mathbb{Z}^d, \Omega \cap 2^{-j}(k + [-L, L]^d) \neq \varnothing\}.$$

Since the support of ϕ is contained in the cube $[-L, L]^d$, it is obvious that if $2^{-j}k \notin \Omega_j$ then $\operatorname{supp} \phi(2^j \cdot - k) \cap \Omega = \varnothing$. Therefore we need only consider k, j's with $2^{-j}k \in \Omega_j$, $j \in \mathbb{N}$.

To simplify our notation, for each lattice point $\gamma = 2^{-j}k$ of Ω_j, we write ϕ_γ instead of $2^{jd/2}\phi(2^j \cdot - k)$. However, since a lattice point may belong to different Ω_j's we will always correlate ϕ_γ with a specific dyadic level j which we make clear in all instances.

The construction of $V_j(\Omega)$, $j \in \mathbb{N}$, proceeds by partitioning Ω_j into a family $\mathscr{C}_j$ of disjoint subsets (cells) C of Ω_j. In other words, each $\mathscr{C}_j$ consists of a collection of disjoint (cells) $C \subset \Omega_j$ such that $\bigcup_{C \in \mathscr{C}_j} = \Omega_j$.

Of course, not all bounded domains Ω will admit a multiresolution analysis $V_j(\Omega)$, $j \in \mathbb{N}$. The admissibility of Ω depends foremost on the properties of the cells C in $\mathscr{C}_j$. We will briefly recall the notation of [2] and describe the properties imposed on the cells in [2] that guarantee the existence of $V_j(\Omega)$, $j \in \mathbb{N}$. For the reader who will find the ensuing notation rather cumbersome, we want only to mention that any bounded simply connected domain Ω in $\mathbb{R}^2$ whose boundary can be partitioned into coordinate wise Lip 1 curves and satisfies (along with Ω^c) the uniform cone property admits such a multiresolution as was proved in [2].

We assume that each $\mathscr{C}_j$ can be partitioned into subcollections $\mathscr{C}_j(I, \sigma)$ where $I \subset \{1, ..., d\}$ and $\sigma = (\sigma_i)_{i \in I} \subset \{-1, 1\}^{|I|}$, i.e.,

$$\mathscr{C}_j = \bigcup_{I, \sigma} \mathscr{C}_j(I, \sigma),$$

where,

$$\mathscr{C}_j(I, \sigma) \cap \mathscr{C}_j(I', \sigma') = \emptyset, \qquad \text{for} \quad (I, \sigma) \neq (I', \sigma').$$

Moreover, each cell $C \in \mathscr{C}_j(I, \sigma)$ is of the form

$$C = k + D(k),$$

with $k \in \Omega_j$ a lattice point (called the representer of C) and

$$D(k) \subset \operatorname{span}\{e_i : i \in I\} \cap 2^{-j}\mathbb{Z}^d$$

with e_i, $i = 1, ..., d$, the coordinate vectors in $\mathbb{R}^d$. It is further required that $\operatorname{dist}(k, \partial\Omega) \geqslant \text{const}\, 2^{-j}$, where the constant does not depend on j, whenever $I \neq \emptyset$ and that $0 \in D(k)$.

For each cell C and its representer k, we define

$$G(C) := k + 2^{-j} T_\sigma \Lambda, \qquad \Lambda := \{\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}^d, 0 \leqslant \alpha_i \leqslant N, 1 \leqslant i \leqslant d\},$$

where for a sequence $\sigma \in \{-1, 1\}^{|I|}$ the transformation T_σ is defined on $\mathbb{R}^d$ by

$$T_\sigma\left(\sum_{i=1}^{d} \lambda_i e_i\right) := \sum_{i \in I} \sigma_i \lambda_i e_i.$$

In other words $G(C)$ consists of a square array of $(N+1)^{|I|}$ lattice points emanating from k and expanded in the direction defined by I and σ.

Also, for a set $K=\{k_1, ..., k_m\} \subset \{1, ..., d\}$ with $k_1 < k_2 < \cdots < k_m$, and a point $x \in \mathbb{R}^d$, we define x_K to be the point whose coordinates are those of x corresponding to the indices of K, i.e., $x_K := (x_{k_1}, ..., x_{k_m})$. Accordingly, we denote by $\Pi_N(I)$ the space of all polynomials $P(x_I)$ of coordinate degree N. If $C \in \mathscr{C}_j(I, \sigma)$ is a cell in $\mathscr{C}_j$, we let $\Lambda(C)$ be the set of all $\alpha \in \Lambda$ for which $\alpha_j = 0$, $j \in \{1, ..., d\} \backslash I$.

Further assumptions have to be made on the cells in order to ensure that the representer of each cell is located well inside the domain Ω, and to guarantee the nestedness of the sequence $V_j(\Omega)$, $j \in \mathbb{N}$, and the existence of a biorthogonal dual basis whose support is sufficiently small. We are not going to describe these additional assumptions (although we impose them) since they are amply reported on in [2].

We are ready now to describe a biorthogonal basis for $V_j(\Omega)$, $j \in \mathbb{N}$. We let P_ν, $\nu \in G(C)$, denote the Lagrange polynomials in $\Pi_N(I)$ which are defined by the interpolation conditions

$$P_\nu(\gamma) = \delta_{\nu, \gamma}, \qquad \gamma, \nu \in G(C). \tag{3.8}$$

It is apparent that the polynomials P_ν, $\nu \in G(C)$, form a basis for $\Pi_N(I)$. For each $\nu \in G(C)$, we define

$$\Phi_\nu := \sum_{\gamma \in C} P_\nu(\gamma)\, \phi_\gamma, \qquad \nu \in G(C).$$

A basis for $V_j(\Omega)$ is given by the set of all functions

$$\Phi_\nu, \qquad \nu \in G_j := \bigcup_{C \in \mathscr{C}_j} G(C).$$

As was proved in [2, Proposition 3.3], the functions $\tilde{\Phi}_\gamma$ defined by

$$\tilde{\Phi}_\gamma :=\tilde{\phi}_\gamma, \quad \gamma \in G_j,$$

constitute a dual system to Φ_γ, $\gamma \in G_j$, i.e.,

$$\langle \Phi_\gamma, \tilde{\Phi}_{\gamma'} \rangle_\Omega := \int_\Omega \Phi_\gamma(x)\, \overline{\tilde{\Phi}'_\gamma(x)}\, dx = \delta_{\gamma, \gamma'}, \qquad \gamma, \gamma' \in G_j.$$

For each $j \in \mathbb{N}$, $V_j(\Omega)$ is defined as the linear span of the functions Φ_γ, $\gamma \in G_j$, and we have

$$V_j(\Omega) \subset V_{j+1}(\Omega) \qquad \text{and} \qquad \overline{\bigcup_{j \in \mathbb{N}} V_j(\Omega)} = L_2(\Omega).$$

For $j \in \mathbb{N}$ we define on $L_1(\Omega)$ the operators

$$Q_j f := \sum_{\gamma \in G_j} \langle f, \tilde{\Phi}_\gamma \rangle_\Omega \, \Phi_\gamma. \tag{3.9}$$

It is easily seen that Q_j, $j \in \mathbb{N}$, are uniformly bounded projectors from $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$, onto $V_j(\Omega)$ (see [2]). We intend to characterize Besov spaces on Ω in terms of the errors of approximation induced by Q_j, $j \in \mathbb{N}$.

4. CHARACTERIZATIONS OF BESOV SPACES BY THE PROJECTORS Q_j, $j \in \mathbb{N}$.

Our goal in this section is to derive an analogue of Theorem 2.30 for the operators Q_j, $j \in \mathbb{N}$, on a domain Ω that admits a multiresolution analysis $V_j(\Omega)$, $j \in \mathbb{N}$, as described in the previous section. In particular, we will prove the following theorem:

THEOREM 4.1. *Let Ω be a minimally smooth bounded domain which admits a multiresolution analysis as described in Section* 3. *Let also* $0 < p \leqslant 1$, $\alpha > d(1/p - 1)$, $0 < q \leqslant \infty$, *and* $N \in \mathbb{N}$ *with* $N + 1 > \alpha$. *If ϕ satisfies properties* (P1)–(P5), *then the following quasi-norms are equivalent*:

$$\begin{aligned} &\text{(i)} \quad \|f\|^q_{B^\alpha_q(L_p(\Omega))}, \\ &\text{(ii)} \quad \sum_{j \geqslant -1} 2^{j\alpha q} \, \|f - Q_j f\|^q_{L_p(\Omega)}, \\ &\text{(iii)} \quad \sum_{j \geqslant 0} 2^{j\alpha q} \, \|Q_j f - Q_{j-1} f\|^q_{L_p(\Omega)}, \end{aligned} \tag{4.2}$$

with constants of equivalency independent of $f \in B^\alpha_q(L_p(\Omega))$.

Following verbatim the proof of Theorem 2.30 we need to establish a Jackson type and a Bernstein type inequality. In other words it is sufficient to prove the following theorems:

THEOREM 4.3. *Let Ω be a minimally smooth bounded domain which admits a multiresolution analysis as described in Section* 3. *Assume that ϕ satisfies assumptions* (P1)–(P3) *and let* $(Q_j)_{j \in \mathbb{N}}$ *be the sequence of operators defined by* (3.9). *If* $0 < p \leqslant 1$, *and* $\alpha \in \mathbb{R}$ *satisfies* $N + 1 > \alpha \geqslant d(1/p - 1)$, *then there exists a constant such that for every* $f \in B^\alpha_p(L_p(\Omega))$

$$\|f - Q_j f\|_{L_p(\Omega)} \leqslant \text{const } 2^{-j\alpha} \, \|f\|_{B^\alpha_p(L_p(\Omega))}, \qquad j = 1, 2, \ldots. \tag{4.4}$$

THEOREM 4.5. *Let* $(Q_j)_{j\in\mathbb{N}}$ *be the sequence of operators defined by* (3.9) *and assume that* ϕ *satisfies assumptions* (P1), (P4), (P5). *Then, for every* $f\in L_p(\Omega)$, $k\in\mathbb{N}$, *and* $0<\mu\leqslant p\leqslant 1$,

$$\omega_r(f, 2^{-k})_p^{\mu}\leqslant \text{const}\, 2^{-kr\mu}\sum_{j=-1}^{k} 2^{jr\mu}\,\|f-Q_j(f)\|_{L_p(\Omega)}^{\mu}, \tag{4.6}$$

where $Q_{-1}:=0$.

Proof of Theorem 4.3. For the proof of the theorem we need only to verify that the operators Q_j, $j\in\mathbb{N}$, satisfy the assumptions of Theorem 2.7. That is, we need to show that for every $j\in\mathbb{N}$, there are extension operators $\mathscr{E}_j$ of Q_j on $\mathbb{R}^d$ such that assumptions (J1)–(J5) are satisfied.

Let $j\in\mathbb{N}$, and $f\in L_1(\mathbb{R}^d)$. We define the extension operators $\mathscr{E}_j$ on $L_1(\mathbb{R}^d)$ by

$$\mathscr{E}_j f:=\sum_{\gamma\in\mathscr{L}_{\Omega_j}}\langle f, \tilde{\Phi}_\gamma\rangle_{\mathbb{R}^d}\,\Phi_\gamma, \tag{4.7}$$

where

$$\mathscr{L}_{\Omega_j}:=G_j\cup(\mathscr{L}_j\backslash\Omega_j),$$

and

$$\Phi_\gamma:=\begin{cases}\phi_\gamma, & \gamma\in\mathscr{L}_j\backslash\Omega_j\\ \Phi_\gamma, & \gamma\in G_j,\end{cases}\qquad \tilde{\Phi}_\gamma:=\begin{cases}\tilde{\phi}_\gamma, & \gamma\in\mathscr{L}_j\backslash\Omega_j\\ \tilde{\Phi}_\gamma, & \gamma\in G_j.\end{cases}$$

It is trivially seen that the operators Q_j, $j\in\mathbb{N}$, satisfy (J4), (J5), with their extension operators $\mathscr{E}_j$, $j\in\mathbb{N}$, as above satisfying (J1), (J2). Thus, we have only to establish that the $\mathscr{E}_j$'s satisfy (J3); that is, for every $j\in\mathbb{N}$,

$$\mathscr{E}_j P=P,\qquad P\in\Pi_N. \tag{4.8}$$

Since $\mathscr{E}_j$ is linear, it is sufficient to prove (4.8) for polynomials $P\in\Pi_N$ of the form $P(x_1, ..., x_d):=P_1(x_1)\cdots P_d(x_d)$ where the P_i's are univariate polynomials. From assumptions (P1)–(P3) of Section 3, it follows that

$$P=\sum_{\gamma\in\mathscr{L}_j}\langle P, \tilde{\phi}_\gamma\rangle\,\phi_\gamma,$$

and that $\langle P, \tilde{\phi}_\gamma\rangle=2^{-jd/2}P(\gamma)$, $\gamma\in\mathscr{L}_j$.

We recall that for every $C\in\mathscr{C}_j$, say $C\in C_j(I,\sigma)$, there exists a basis $\{P_{C,v}, v\in G(C)\}$ for $\Pi_N(I)$ (the space of all polynomials of coordinate degree N in the coordinate directions of I), defined by (3.8) (we have slightly changed our notation to encode the dependence of the basis on the corresponding cell C). Writing $P=P_I P_{I'}$ where $P_I \in \Pi_N(I)$ and $P_{I'} \in \Pi_N(I')$, $I' := \{1, ..., d\} \backslash I$, it follows that there exist coefficients $\lambda_C(v)$, $v \in G(C)$, such that

$$P_I = \sum_{v \in G(C)} \lambda_C(v)\, P_{C,v}. \tag{4.10}$$

From (4.9) and (4.10) we derive that

$$\begin{aligned} P &= \sum_{\gamma \in \mathscr{L}_j \backslash \Omega_j} 2^{-jd/2} P(\gamma)\, \phi_\gamma + \sum_{\gamma \in \Omega_j} 2^{-jd/2}\, P(\gamma)\, \phi_\gamma \\ &= \sum_{\gamma \in \mathscr{L}_j \backslash \Omega_j} \langle P, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d}\, \Phi_\gamma + 2^{-jd/2} \sum_{C \in \mathscr{C}_j} \sum_{\gamma \in C} P_I(\gamma)\, P_{I'}(\gamma)\, \phi_\gamma \\ &= \sum_{\gamma \in \mathscr{L}_j \backslash \Omega_j} \langle P, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d}\, \Phi_\gamma + 2^{-jd/2} \sum_{C \in \mathscr{C}_j} \sum_{\gamma \in C} \sum_{v \in G(C)} P_{I'}(\gamma)\, \lambda_C(v)\, P_{C,v}(\gamma)\, \phi_\gamma. \end{aligned}$$

Taking into account that $P_{I'}(\gamma) =: P_{I', C}$ is constant for all $\gamma \in C$, and that for every $v \in G(C)$, $2^{-jd/2} P_{I', C} \lambda_C(v) = \langle P, \tilde{\Phi}_v \rangle_\Omega$ we get that

$$\begin{aligned} P &= \sum_{\gamma \in \mathscr{L}_j \backslash \Omega_j} \langle P, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d}\, \Phi_\gamma + 2^{-jd/2} \sum_{C \in \mathscr{C}_j} \sum_{v \in G(C)} P_{I', C} \lambda_C(v)\, \Phi_v \\ &= \sum_{\gamma \in \mathscr{L}_j \backslash \Omega_j} \langle P, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d}\, \Phi_\gamma + \sum_{\gamma \in G_j} \langle P, \tilde{\Phi}_\gamma \rangle_\Omega\, \Phi_\gamma \\ &= \sum_{\gamma \in \mathscr{L}_{\Omega_j}} \langle P, \tilde{\Phi}_\gamma \rangle_{\mathbb{R}^d}\, \Phi_\gamma \\ &= \mathscr{E}_j P. \quad \blacksquare \end{aligned}$$

For the proof of Theorem 4.5 we intend to establish conditions (B1)–(B3) of Section 2. Condition (B2) follows from the nestedness of the multiresolution analysis $V_j(\Omega)$, $j \in \mathbb{N}$. As far as (B3) is concerned, we recall that for each $\gamma \in G_j$, $\Phi_\gamma = \sum_{v \in C} P_\gamma(v)\, \phi_v$, where $|P_\gamma(v)|$, $v \in C$, are bounded by an absolute constant. Using that

$$\|\Delta_h^r(\phi_v(\cdot))\|_{L_p} = 2^{jd/2}\, \|\Delta_{2^j h}^r(\phi(2^j \cdot))\|_{L_p},$$

it easily follows that

$$\omega_r(\phi_v, t)_p = 2^{jd/2} 2^{-jd/p} \omega_r(\phi, 2^j t)_p \leqslant \text{const}\, 2^{jd(1/2 - 1/p)} \min(1, (2^j t)^r). \tag{4.11}$$

Therefore, we have

$$\omega_r(\Phi_\gamma, t)_p \leqslant \text{const}\, 2^{jd(1/2-1/p)} \min(1, (2^j t)^r),$$

which is condition (B3).

It remains to prove property (B1). For any cube $Q \subset \mathbb{R}^d$, we let $\lambda(Q)$ denote the set of $\gamma \in \mathscr{L}_j$, such that ϕ_γ is not identically zero on Q while by $\Lambda(Q)$, we denote the family of all $\gamma \in G_j$ such that Φ_γ does not vanish identically on Q.

Since different quasi-norms on finite-dimensional spaces are equivalent, the local linear independence of the shifts of ϕ implies that for any dyadic cube Q of side length 2^{-j} and any sequence of complex numbers $b := (b(\gamma))_{\gamma \in \lambda(Q)}$

$$\|b\|_{l_p(\lambda(Q))} \approx 2^{jd(1/p-1/2)} \left\| \sum_{\gamma \in \lambda(Q)} b(\gamma)\, \phi_\gamma \right\|_{L_p(Q)}, \tag{4.12}$$

for some constants independent of Q and $j \in \mathbb{N}$.

We now fix $\gamma \in G(C)$, $C \in C_j$, $j \in \mathbb{N}$, and we consider the basis functions

$$\Phi_\gamma = \sum_{\nu \in C} P_\gamma(\nu)\, \phi_\nu$$

with P_γ the Lagrange polynomials defined in (3.8). We note that $P_\gamma(\gamma) = 1$ while $|P_\gamma(\nu)| \leqslant \text{const}$, $\nu \in C$, for some constant depending only on ϕ. From (P1) we know that for each $\nu \in G_j$, ϕ_ν does not vanish identically on the dyadic cube Q^ν with sidelength 2^{-j} and lower left corner at ν. Moreover, all the Q^ν, $\nu \in G_j$, are contained in Ω. The linear independence of the shifts of ϕ guarantees that $\gamma \in \Lambda(Q^\gamma)$, i.e., that Φ_γ does not vanish identically on Q^γ. It follows from (4.12) that

$$\text{const}\, 2^{jd(1/2-1/p)} \leqslant \|\Phi_\gamma\|_{L_p(Q^\gamma)} \leqslant \|\Phi_\gamma\|_{L_p(\mathbb{R}^d)} \leqslant \text{const}\, 2^{jd(1/2-1/p)}, \tag{4.13}$$

where we have used the fact that each cell has a number of lattice points which is bounded independently of j and the cell (see [2, Theorem 2.18]).

LEMMA 4.14. *Let $\mu \in G_j$ and Q^μ, $\Lambda(Q^\mu)$ be as described above. If $0 < p \leqslant \infty$ then for any sequence $(b(\gamma))_{\gamma \in \Lambda(Q^\mu)}$, we have*

$$\left\| \sum_{\gamma \in \Lambda(Q^\mu)} b(\gamma)\, \Phi_\gamma \right\|_{L_p(Q^\mu)} \geqslant \text{const}\, 2^{jd(1/2-1/p)}\, |b(\mu)|. \tag{4.15}$$

Proof. We fix an arbitrary $\mu \in G_j$. It follows from the definitions that on Q^μ we have

$$\begin{aligned}\sum_{\gamma \in \Lambda(Q^\mu)} b(\gamma)\, \Phi_\gamma &= \sum_{C \in C_j} \sum_{\gamma \in G(C) \cap \Lambda(Q^\mu)} b(\gamma)\, \Phi_\gamma \\ &= \sum_{C \in C_j} \sum_{\gamma \in G(C) \cap \Lambda(Q^\mu)} b(\gamma) \sum_{\nu \in C} P_\gamma(\nu)\, \phi_\nu \\ &= \sum_{C \in C_j} \sum_{\nu \in C} \left(\sum_{\gamma \in G(C) \cap \Lambda(Q^\mu)} b(\gamma)\, P_\gamma(\nu) \right) \phi_\nu \\ &= \sum_{C \in C_j} \sum_{\nu \in C} c(\nu)\, \phi_\nu = \sum_{\nu \in \Omega_j} c(\nu)\, \phi_\nu,\end{aligned}$$

where

$$c(\nu) := \sum_{\gamma \in G(C) \cap \Lambda(Q^\mu)} b(\gamma)\, P_\gamma\, (\nu), \qquad \nu \in \Omega_j. \tag{4.16}$$

Therefore, we have that

$$\sum_{\gamma \in \Lambda(Q^\mu)} b(\gamma)\, \Phi_\gamma = \sum_{\nu \in \lambda(Q^\mu)} c(\nu)\, \phi_\nu, \qquad \text{on } Q^\mu. \tag{4.17}$$

Finally, taking the L_p norm of (4.17) over Q^μ and using (4.12), we get

$$\begin{aligned}\left\| \sum_{\gamma \in \Lambda(Q_\mu)} b(\gamma)\, \Phi_\gamma \right\|_{L_p(Q^\mu)} &\geqslant \text{const } 2^{jd(1/2 - 1/p)}\, \|c(\nu)\|_{l_p(\lambda(Q^\mu))} \\ &\geqslant \text{const } 2^{jd(1/2 - 1/p)}\, |c(\mu)|.\end{aligned}$$

The result now follows because $c(\mu) = b(\mu)$. ∎

We are ready to prove the stability of the basis $\{\Phi_\gamma : \gamma \in G_j\}$.

THEOREM 4.18. *For each $0 < p \leqslant \infty$, the basis $\{\Phi_\gamma : \gamma \in G_j\}$ is uniformly L_p-stable; i.e., there exist positive constants independent of $j \in \mathbb{N}$ such that for any sequence $b := (b(\gamma))_{\gamma \in G_j}$*

$$\|b_{l_p(G_j)} \approx 2^{jd(1/p - 1/2)} \left\| \sum_{\gamma \in G_j} b(\gamma)\, \Phi_\gamma \right\|_{L_p(\Omega)}. \tag{4.19}$$

Proof. Let $g \in V_j(\Omega)$, say $g = \sum_{\gamma \in G_j} b(\gamma)\, \Phi_\gamma$. To establish half of (4.19) we note that by the previous lemma

$$\int_\Omega |g|^p \geqslant \sum_{\mu \in G_j} \int_{Q^\mu} \left| \sum_{\gamma \in \Lambda(Q^\mu)} b(\gamma)\, \Phi_\gamma \right|^p$$
$$\geqslant \text{const}\; 2^{jdp(1/2-1/p)} \sum_{\mu \in G_j} |b(\mu)|^p$$
$$= \text{const}\; 2^{jdp(1/2-1/p)} \|b\|^p_{l_p(G_j)}.$$

For the other inequality, we note that for each $x \in \Omega$ the number of Φ_γ which are not zero at x is bounded inependently of x. Therefore,

$$|g(x)|^p \leqslant \text{const} \sum_{\gamma \in G_j} |b(\gamma)|^p\, |\Phi_\gamma(x)|^p.$$

The proof concludes by integrating the last inequality over Ω and using (4.13). ∎

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