1.) Let M^2 $\{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ be the punctured unit disc. Let (r,θ) be polar coordinates on M. Define a metric on M by

$$(\partial_r, \partial_r) := \frac{1}{(1-r^2)^2}, \quad (\partial_r, \partial_\theta) := 0, \quad (\partial_\theta, \partial_\theta) := \frac{r^2}{(1-r^2)^2},$$

where ∂_r and ∂_θ denote the coordinate vector fields. Compute the Gauss curvature of this metric.

- $(1-r^2)\partial_r = e_1$ & $\frac{1-r^2}{r}\partial_\theta = e_2$ is an orthonormal frame $\Rightarrow$ $\omega^1 = \frac{dr}{1-r^2}$ and $\omega^2 = \frac{r\,d\theta}{1-r^2}$ is an orthonormal coframe.

- Compute $\theta^1_2 = a\omega^1 + b\omega^2 = -\theta^2_1$ via $d\omega^1 = -\theta^1_2 \wedge \omega^2$ and $d\omega^2 = -\theta^2_1 \wedge \omega^1$.

* $d\omega^1 = 0 \Rightarrow a = 0$

* $d\omega^2 = \frac{1}{(1-r^2)^2}(1+r^2)\, dr \wedge d\theta = \frac{1+r^2}{r}\omega^1 \wedge \omega^2$

$\Rightarrow b = -\frac{1+r^2}{r} = -\frac{1}{r} - r$

$\Rightarrow \theta^1_2 = -\frac{1+r^2}{1-r^2}\, d\theta$

Extra space for 1).

- $$K\omega^1 \wedge \omega^2 = d\theta^1_2 = \left[\frac{-2r}{1-r^2} - \frac{2r(1+r^2)}{(1-r^2)^2} \right] dr \wedge d\theta$$

$$= \frac{-2r + 2r^3 - 2r - 2r^3}{(1-r^2)^2} \, dr \wedge d\theta$$

$$= -4\omega^1 \wedge \omega^2 \quad \Rightarrow \quad \boxed{K = -4}$$

2.) Let $\omega^1, \dots, \omega^k$ be linearly independent smooth 1-forms on M^n. Suppose that $\alpha_1, \dots, \alpha_k$ are smooth 1-forms and $0 = \alpha_i \wedge \omega^i$. Prove that there exist $h_{ij} = h_{ji} \in C^\infty(M)$ such that $\alpha_i = h_{ij}\omega^j$, $1 \le i, j \le k$.

Given $p \in M$, $\exists$ a nbd U of p on which $\{\omega^1, \dots, \omega^k\}$ can be completed to a local coframe $\{\omega^1, \dots, \omega^k, \dots, \omega^n\}$.

Then on U $\exists$ $h_{ij} \in C^\infty(U)$ s.t.

$$\alpha_i = \sum_{j=1}^{n} h_{ij}\omega^j \qquad 1 \le i \le k$$

Then

$$0 = \alpha_i \wedge \omega^i = \sum_{i=1}^{k} \sum_{j=1}^{n} h_{ij}\omega^i \wedge \omega^j$$

$$= \sum_{i,j=1}^{k} h_{ij}\,\omega^i \wedge \omega^j + \sum_{i=1}^{k} \sum_{j=k+1}^{n} h_{ij}\,\omega^i \wedge \omega^j$$

$$\Rightarrow \quad h_{ij} = 0 \quad \forall\, j > k$$

$$h_{ij} - h_{ji} = 0 \quad \forall\, i,j \le k.$$

So the desired h_{ij} exist on U. From the facts that $\{\alpha_i\}_{i=1}^{k}$ & $\{\omega^i\}_{i=1}^{k}$ are globally defined, we conclude that $\{h_{ij}\}$ are global. ∎

3.) Define $F : \mathbb{R}^2 \to \mathbb{R}^4$ by $(s,t) \mapsto (s^3, s^2t, st^2, t^3)$. Let (w,x,y,z) be coordinates on $\mathbb{R}^4$ and compute the pull-back $F^*(\mathrm{d}w + e^z \mathrm{d}x)$.

$$F^*(e^z) = e^{t^3}$$

$$F^* dw = d s^3 = 3s^2 ds$$

$$F^* dx = d s^2t = 2st\, ds + s^2 dt$$

So $F^*(dw + e^z dx) =$

$$(3s^2 + 2st e^{t^3})\, ds + s^2 e^{t^3}\, dt$$