$\boxed{\text{Ex } 1.9.4}$ $I^n = I \cap \Omega^n(\Sigma_1)$.

$$V_n(I,\Omega)_x = \left\{ E \in G_n(T_xM) : \Omega_{|E} \neq 0,\ \alpha_{|E} = 0 \ \forall \alpha \in I \right\}.$$

Asm. $\alpha_{|E} = 0 \ \forall \alpha \in I^n$.

Q: Is it true that $E \in V_n(I)_x$?

A: Suffices to work w/ homog $\beta \in I$. ✓ $\bigoplus_k I^k$.

If $\beta \in I^k$ $k \geq n$ then $E_{|\beta} = 0$.

Spse $\beta \in I^k$ $k < n$.

If $\beta_{|E} \neq 0$ then $\exists$ $\gamma \in \Omega^{n-k}(\Sigma)$ s.t.

$\beta \wedge \gamma_{|E} \neq 0$. But $\beta \wedge \gamma \in I^n$. So $\times$.

YES.

$\boxed{\text{Ex } 1.9}$ $V \in T_pM$. $V = (v^1, \dots, v^n)$

$u = (u^1, \dots, u^n) = Vt$ in coords.

$u(t)$ $u(0) = p$ $u'(0) = V$.

$$t \mapsto u(t) \mapsto f\circ(u(t)) \mapsto f^i[u(t)] \in \mathbb{R}$$

$\mathbb{R}$ $\quad$ M $\quad$ N $\quad$ $\mathbb{R}$.

$f(p) = f[u(0)]$

$$\frac{df^i}{dt} = \frac{\partial f^i}{\partial u^j} \dot{u}^j \qquad \left.\frac{df^i}{dt}\right|_0 = v^j \frac{\partial f^i}{\partial u^j}(0). \quad = 0 \text{ in coords.}$$

$$\frac{d^2 f^i}{dt^2} = \frac{\partial^2 f^i}{\partial^2 u^k u^j} \dot{u}^j \dot{u}^k \qquad \frac{df^i}{dt}(0) = v^j v^k \frac{\partial^2 f}{\partial u^k \partial u^j}(0).$$

View $\frac{d^k f^i}{dt^k}(0)$ and $\frac{d^k g^i}{dt^k}(0)$ as hom. polys in v^i.

Then f and g have the same k-jet $\Leftrightarrow$ partial derivatives agree up to order k.

$\boxed{\text{Ex. } 19.7.}$

(a) $\{x^1, x^2, x^3, u, p_1, p_2, p_3, p_{11}, p_{12}, p_{13}, p_{22}, p_{23}, p_{33}\}$

$$\dim = \dim \mathbb{R}^3 + \dim \mathbb{R} + 9. = \boxed{13}$$

(b) $\{x^1, x^2, u^1, u^2, p_1, p_2, q_1, q_2, p_{11}, p_{12}, p_{22}, q_{11}, q_{12}, q_{22},$
$p_{111}, p_{112}, p_{122}, p_{222}, q_{111}, q_{112}, q_{122}, q_{222}\}$

$$\dim = \dim \mathbb{R}^2 + \dim \mathbb{R}^2 + 18. = \boxed{22}$$

(c) $J^k(\mathbb{R}^n, \mathbb{R}^m)$. $\quad x = (x^1, \dots, x^n) \in \mathbb{R}^n \quad u = (u^1, \dots, u^m) \in \mathbb{R}^m$

$$J^k(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \times \left(\prod_{l=1}^{k} \mathbb{R}^m \otimes \mathrm{Sym}^l(\mathbb{R}^n)^* \right)$$

So $\dim J^k(\mathbb{R}^n, \mathbb{R}^m) = n + m + m \sum_{l=1}^{k} \dim \mathrm{Sym}^l(\mathbb{R}^n)^*$

$$= n + m + m \sum_{l=1}^{k} \binom{n+l-1}{l}$$

$\boxed{\text{Ex } 1.9.9.}$

$$I = \{x^1 dx^2, dx^3\}_{diff} = \{x^1 dx^2, dx^3, dx^1 \wedge dx^2\}_{alg}$$

$$V_1(I) = \{ v^i \partial_{x^i} : \alpha(v) = 0 \ \forall \alpha \in I \}.$$

at $(1,1,1)$ $\quad \alpha_1 = dx^2 \quad \alpha_2 = dx^3 \quad \beta = dx^1 \wedge dx^2$

$$V_1(I)_{(1,1,1)} = \text{span}\{\partial_{x^1}\}$$

$$V_1(I)_{(0,0,0)} = \cancel{\text{span}\{\partial_{x^1}, \partial_{x^2}\}} = \overset{\text{span } Z}{\{a\partial_{x^1} + b\partial_{x^2} : a, b \text{ not both zero}\}}$$

Note if $E \in V_k(I)$, then each subspace $\tilde{E} \subset E$ of dim $k-1$ is in $V_{k-1}(I)$.

$$V_2(I)_{(1,1,1)} = \phi$$

$$V_2(I)_{(0,0,0)} = \phi$$

At best $\partial_{x^1} \wedge \partial_{x^2} = E \in V_2(I)_{(0,0,0)}$
But $dx^1 \wedge dx^2|_E \neq 0$.

$$V_1(I, \Omega)_{(1,1,1)} = \begin{cases} \phi & a = 0 \\ \text{span}\{\partial_{x^1}\} & a \neq 0 \end{cases}$$

$$\Omega(u\partial_{x^1} + v\partial_{x^2}) = au + bv$$

$$V_1(I, \Omega)_{(0,0,0)} = \{ E = \text{span}\{u\partial_{x^1} + v\partial_{x^2}\} \mid au + bv \neq 0 \}$$

$\boxed{\text{Ex } 1.9.16 \text{ (c)}}$ However $[\partial y^1 \wedge \cdots \wedge \partial y^n]$ is an int. element at every pt.

<u>Rmk</u> (a) is false (unless $y = 0$)

$$E = \partial_{x^1} \wedge \cdots \wedge \partial_{x^2} \text{ an integral element}$$

$$\Rightarrow \quad \theta(\partial_{x^i}) = y^i = 0.$$

(b) is clear

Write $v_i = a_i \partial_z + b_i{}^j \partial_{x^j} + c_i{}^j \partial_{y^j}$.

$$\theta(v_i) = a_i - \sum_j y^j b_i{}^j$$

$$d\theta = \sum_i dx^i \wedge dy^i$$

$$d\theta(v, \cdot) = 0 \iff v = \lambda \partial_z.$$

$$\alpha_i(\cdot) := d\theta(v_i, \cdot).$$

Asm $v^1 \wedge \cdots \wedge v^{n+1} =: E$ is an $(n+1)$-dim'l int. element of I.

Then $\alpha_i|_E = 0$. This implies the α_i are lin. dep. ('cos $\dim E = n+1$, and $\dim E^\perp = n$.)

Wolog $\alpha_{n+1} = t^i \alpha_i \quad 1 \le i \le n.$

$$\Rightarrow v^{n+1} - t^i v_i = \lambda \partial_z.$$

Hence $E = v_1 \wedge \cdots \wedge v_n \wedge \partial_z$. But $\theta(\partial_z) = 1 \neq 0$.

So $\theta|_E \neq 0$

$\Rightarrow E$ is not an int. elm.