DIFFERENTIAL GEOMETRY HW

C. ROBLES

Contents

1. IMPLICIT FUNCTION THEOREM

In lecture we proved the following.

Theorem. *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is a smooth function. Let* $a \in \mathbb{R}$ *and assume that* $\mathrm{d}f(p) \neq 0$ *for all* $p \in f^{-1}(a)$. *Then* $f^{-1}(a)$ *is a manifold.*

The result may be generalized as follows.

Theorem. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map of constant rank k. Let $q \in \mathbb{R}^m$. Then $F^{-1}(q)$ is a manifold.

HW 1.1. Prove the generalization.

2. MANIFOLDS

HW 2.1 (Product manifolds)**.** Assume that M and N are (differentiable) manifolds of dimensions m and n respectively. Prove that $P = M \times N$ is a manifold of dimension $m+n$.

HW 2.2 (Surfaces of revolution)**.** Consider $\mathbb{R}^3$ with coordinates (x, y, z). Let $\gamma(t) = (x(t), 0, z(t))$ be a curve in the xz-plane with the properties that $z(t) > 0$ and $\dot{z}(t) > 0$ for all t. Let M be the surface of revolution obtained by revolving the image of γ about the z-axis. Prove that M is a manifold.

2.1. **Spheres.**

Let $x = (x^1, \ldots, x^{n+1})$ denote a point on the n-sphere S^n. In lecture we defined an atlas $\mathcal{A} = \{(U_j^\pm, \varphi_j^\pm)\}_{j=1}^{n+1}$ on the n-sphere

$$S^n \ = \ \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}\,.$$

Specifically, $U_j^+ = \{x \in S^n \mid x^j > 0\}$ and $U_j^- = \{x \in S^n \mid x^j < 0\}$. The coordinates $\varphi_j^\pm : U_j^\pm \to D^n = \{u \in \mathbb{R}^n \mid |u| < 1\}$ are simply projection $\varphi_j^\pm(x) = (x^1, \ldots, \widehat{x^j}, \ldots, x^{n+1})$.

In this exercise you will construct an atlas $\mathcal{A}' = \{(U, \varphi), (V, \psi)\}$ on the n-sphere that consists of only two coordinate charts. Define the North and South poles to be

$$p = (0, \ldots, 0, 1) \quad \text{and} \quad q = (0, \ldots, 0, -1)\,.$$

Set

$$\begin{aligned} U &= S^n \backslash \{p\} \\ V &= S^n \backslash \{q\}\,. \end{aligned}$$

Give $\mathbb{R}^{n+1}$ coordinates $x = (x^1, \ldots, x^{n+1})$.

Define stereographic projection (from the North pole p)

$$\varphi : U \to \mathbb{R}^n = \{x^{n+1} = 0\} \subset \mathbb{R}^{n+1}$$

as follows. Given $u \in U$, let $L(u)$ be the unique line through u and p. Note that this line intersects the n-plane $\{x^{n+1} = 0\}$ in exactly one point. Define $\varphi(u) = L(u) \cap \{x^{n+1} = 0\}$ to be this point of intersection.

Similarly, define stereographic projection (from the South pole q)

$$\psi : V \to \mathbb{R}^n = \{x^{n+1} = 0\} \subset \mathbb{R}^{n+1}\,.$$

HW 2.3. Prove $\mathcal{A}'$ is an atlas on S^n.

HW 2.4. Prove that the two atlases $\mathcal{A}$ and $\mathcal{A}'$ are compatible. (We say *$\mathcal{A}$ and $\mathcal{A}'$ determine the same $\mathcal{C}^\infty$ structure on S^n.*)

2.2. **Embedding $\mathbb{RP}^2$ in $\mathbb{R}^4$.** Define $F : \mathbb{R}^3 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$, $(x, y, z) = p \in \mathbb{R}^3$. Let $\varphi : S^2 \to \mathbb{R}^4$ denote the restriction of F to the 2-sphere $S^2 \subset \mathbb{R}^3$. Notice that $\varphi(p) = \varphi(-p)$, so that φ induces a well-defined $\tilde{\varphi} : \mathbb{RP}^2 \to \mathbb{R}^4$ by $\tilde{\varphi}([p]) = \varphi(p)$, with $[p]$ the equivalence class of $\pm p \in S^2$.

HW 2.5. Prove that $\tilde{\varphi}$ is an immersion.

HW 2.6. Prove that $\tilde{\varphi}$ is injective.

HW 2.7. Using the fact that $\mathbb{RP}^2$ is compact, prove that $\tilde{\varphi}$ is an embedding.

3. The tangent space

Notation. $\partial_{x^j} := \dfrac{\partial}{\partial x^j}$.

Definition. Given an n-dimensional manifold M and $p \in M$, let $\mathcal{C}^\infty_p = \{(f : U \to \mathbb{R}) \in \mathcal{C}^\infty \mid p \in U \subset M \text{ open}\}$ denote the *smooth functions at p*. A *derivation at p* is a map $X_p : \mathcal{C}^\infty_p \to \mathbb{R}$ satisfying the following two properties:

Linearity: $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$, for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathcal{C}^\infty_p$.

Leibniz: $X_p(fg) = X_p(f)\, g(p) + f(p)\, X_p(g)$.

The *tangent space T_pM at p* is the set of derivations at p.

HW 3.1. Prove that T_pM is a vector space.

Definition. Let $F : M \to N$ be a smooth map between manifolds. The *push-forward* by F at $p \in M$ is a map $F_* : T_pM \to T_{F(p)}N$ defined as follows. Given $g \in \mathcal{C}^\infty_{F(p)}$, $F_*(X_p)g := X_p(g \circ F)$.

HW 3.2. Prove that $F_*(X_p) \in T_{F(p)}N$.

HW 3.3. Prove that $F_* : T_pM \to T_{F(p)}N$ is a linear map.

HW 3.4. Let $F : M \to N$ be a smooth map between manifolds, $\dim M = m$ and $\dim N = n$. Let $x : U \to \mathbb{R}^m$ be a coordinate chart on M and $y : V \to \mathbb{R}^n$ a coordinate chart on N. Assume that $F(U) \subset V$.

We have seen in class that the ∂_{x^j}, $j = 1, \ldots, m$ span T_xM; and the ∂_{y^s}, $s = 1, \ldots, n$ span T_yN. Above you proved that $F_* : T_xM \to T_{F(x)}N$ is a linear map. Identify the matrix representing this linear map with respect to the given bases.

HW 3.5. Recall the coordinates on S^n induced by stereographic projection in §**??**. Consider the first coordinate chart in the atlas $(U, \varphi = y)$, $y \in \mathbb{R}^n$. Express the push-forward

$$\varphi^{-1}_*(\partial_{y^j})_{y=0} \in T_qS^n$$

($q = (0, \ldots, 0, -1)$ is the South pole) of ∂_{y^j} at $y = 0 \in \mathbb{R}^n$ as a vector in $\mathbb{R}^{n+1} \supset S^n$, $j = 1, \ldots, n$.

More generally, if $y_0 \in \mathbb{R}^n$, then $\varphi^{-1}_*(\partial_{y^j})_{y=y_0}$ is tangent to S^n at some point $x \in S^n$. What is the point x? Express the push-forward as a vector in $\mathbb{R}^{n+1}$.

HW 3.6. Consider the coordinates $U = \{[s:t] \in \mathbb{P}^1 \mid s \neq 0\}$ and $\varphi : U \to \mathbb{R}$, $\varphi([s:t]) = t/s$ on $\mathbb{P}^1$. Let x denote coordinates on $\mathbb{R}$. Set $X_{[1:x]} = \varphi^{-1}_*(\partial/\partial x) \in T_{[1:x]}\mathbb{P}^1$.

Recall the Veronese map $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ sending $[s:t] \mapsto [s^2 : st : t^2]$. The image $\nu_2(U) \subset V = \{[u:v:w] \in \mathbb{P}^2 \mid u \neq 0\}$. Define coordinates $\psi : V \to \mathbb{R}^2$ by $\psi([u:v:w]) = (\frac{v}{u}, \frac{w}{u})$ and let $y = (y^1, y^2)$ be coordinates on $\mathbb{R}^2$. Set $Y_{j,[1:y^1:y^2]} = \psi^{-1}_*(\partial_{y^j}) \in T_{[1:y^1:y^2]}\mathbb{P}^2$, $j = 1, 2$.

(a) Express the push forward $(\nu_2)_* X_{[1:x]}$ in terms of the $Y_{1,[1:x:x^2]}$ and $Y_{2,[1:x:x^2]}$.

(b) Prove that ν_2 is an embedding.

Definition. A vector field X on a M is *smooth* if for every smooth function $f : M \to \mathbb{R}$, the function $Xf : M \to \beta$, mapping $p \mapsto X_p f$, is smooth.

HW 3.7. Let X be a vector field on a manifold M, $\dim M = n$. Within a local coordinate chart $x : U \subset M \to \mathbb{R}^n$, X may be expressed as $X = f^i \frac{\partial}{\partial x^i}$ for some functions $f^i : U \to \mathbb{R}$. (Recall, that the $\frac{\partial}{\partial x^j}$, $j = 1, \ldots, n$, span $T_x M$.) Prove that X is smooth on U if and only if the f^i are smooth functions.

HW 3.8. Prove that a vector field X is smooth, if and only if the map $X : M \to TM$, sending $p \mapsto X_p \in T_p M$, is smooth as a map between manifolds.

Definition. Let $F : M \to N$ be a smooth map between manifolds, and $X_p \in T_p M$. The *push-forward* of X_p by F is the derivation $F_*(X_p) \in T_{F(p)}N$ defined by $F_*(X_p)g := X_p(g \circ F)$ for all $g \in \mathcal{C}^\infty_{F(p)}$.

Remark. The push-forward is a map $T_p M \to T_{F(p)}N$. It is *not* necessarily true that F_* maps vector fields on M to vector fields on N. For example, suppose that F is not injective and that X is a vector field on M. Then $q = F(p_1) = F(p_2)$ for some $p_1 \neq p_2 \in M$. If $F_*(X_{p_1}) \neq F_*(X_{p_2}) \in T_q N$, then $F_* X$ is not well-defined at q.

HW 3.9. Assume that X is a smooth vector field on M, and that $F : M \to N$ is an embedding. Prove that the push-forward $F_* X$ is a smooth vector field on $F(M) \subset N$.

HW 3.10. Prove that the Lie bracket of two smooth vector fields is a smooth vector field.

HW 3.11. Given two smooth vector fields X, Y on a manifold M, and two smooth functions f, $g : M \to \mathbb{R}$, note that fX and gY are also smooth vector fields on M. Prove that

$$[fX, gY] \;=\; fg\,[X,Y] \;+\; f\,(Xg)\,Y \;-\; g\,(Yf)\,X\,.$$

HW 3.12. Let $F : M \to N$ be a smooth map of manifolds. Prove that $F_* : TM \to TN$ is a smooth map of manifolds.

4. Forms

HW 4.1. Prove that the cotangent bundle T^*M is a smooth manifold.

Definition. A 1-form σ on a manifold M is *smooth* if, given any smooth vector field X on M, the assignment $p \mapsto \sigma_p(X_p)$, $p \in M$, is a smooth function on M.

HW 4.2. Let σ be a 1-form on a manifold M, and (U, x) a local coordinate chart on M. Since the $\{\mathrm{d}x^i\}_{i=1}^{\dim M}$ span the cotangent space $T^*_x M$ we can write $\sigma = \sigma_i(x)\ \mathrm{d}x^i$ for some functions $\sigma_i(x) : U \to \mathbb{R}$. Prove that σ is smooth on U if and only if the σ_i are smooth functions.

HW 4.3. Let V be a vector space. Prove that a k-linear functional $L \in V^* \otimes \cdots \otimes V^* = (V^*)^{\otimes k}$ on V is skew if and only if $L(v_1, \ldots, v_k) = 0$ whenever some $v_i = v_j$, $i \neq j$.

HW 4.4. Let $\alpha_1, \ldots, \alpha_k \in V^*$. Prove that $\alpha_1 \wedge \cdots \wedge \alpha_k \neq 0$ if and only if the $\{\alpha_i\}_{i=1}^k$ are linearly independent.

Definition. A k-form ω on M is *smooth* if, whenever $X_1, \ldots, X_k$ are smooth vector fields on M, the function $M \to \mathbb{R}$ defined by $p \mapsto \omega_p(X_1, \ldots, X_k)$, is smooth.

HW 4.5. Given a coordinate chart (U, x) on M, prove that the coordinate k-forms $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ are smooth on U.

HW 4.6. The coordinate k-forms $\{\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}{}_{|p} \mid 1 \leq i_i < \cdots < i_k \leq \dim M\}$ are a basis of $\bigwedge^k T_p^* M$, $p \in U$. So locally any k-form ω may be expressed as $\omega = \omega_{i_1 \cdots i_k}(x) \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ where the $\omega_{i_1 \cdots i_k}(x)$ are functions on U. Prove that ω is smooth on U if and only if the $\omega_{i_1 \cdots i_k}(x)$ are smooth functions.

Definition. Let M be an n-dimensional manifold. A *local framing of* M is a collection of n (smooth) vector fields $X_1, \ldots, X_n$ defined on an open set $U \subset M$ with the property that $X_1(p), \ldots, X_n(p)$ is a basis of $T_p M$ for each $p \in U$.

A *local coframing of* M is a collection of n (smooth) 1-forms $\alpha^1, \ldots, \alpha^n$ defined on an open set $U \subset M$ with the property that $\alpha^1_p, \ldots, \alpha^n_p$ is a basis of $T^*_p M$ for each $p \in U$.

HW 4.7. Let $X_1, \ldots, X_n$ be a smooth local framing of M defined on $U \subset M$. Given $p \in M$ define $\xi^1_p, \ldots \xi^n_p \in T^*_p M$ by $\xi^j_p(X_k(p)) = \delta^j_k$. Prove that the $\xi^1, \ldots, \xi^n$ form a (smooth) local coframing on M.

HW 4.8. Let V be a vector space and $\varphi_i \in \bigwedge^1 V = V^*$, $i = 1, \ldots, k$. Given $v_i \in V$, prove that

$$\varphi_1 \wedge \cdots \wedge \varphi_k(v_1, \ldots, v_k) \;=\; \det\big(\varphi_i(v_j)\big)\,.$$

Definition. Let $F : M \to N$ be a smooth map of manifolds. The *pull-back by* F is a map $F^* : \bigwedge^k T^*_{F(p)} N \to \bigwedge^k T^*_p M$ defined as follows. Let $\alpha \in \bigwedge^k T^*_{F(p)} N$ and $X_1, \ldots, X_k \in T_p M$. Then

$$(F^*\alpha)(X_1, \ldots, X_k) \;:=\; \alpha(F_* X_1, \ldots, F_* X_k)\,.$$

HW 4.9. Prove that $F^* : \bigwedge^k T^*_{F(p)} N \to \bigwedge^k T^*_p M$ is a linear map.

HW 4.10. Given $\alpha \in \bigwedge^k T^*_{F(p)} N$ and $\beta \in \bigwedge^\ell T^*_{F(p)} N$, prove that $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$.

Definition. Let V be a vector space and $v \in V$. *Interior multiplication by* v (or *lefthook by* v) is a map

$$i_v \;=\; v \lrcorner \;:\; \textstyle\bigwedge^r V \to \bigwedge^{r-1} V$$

defined as follows. Given $\varphi \in \bigwedge^r V$ and $v_1, \ldots, v_{r-1} \in V$

$$(i_v \varphi)(v_1, \ldots v_{r-1}) \;:=\; \varphi(v, v_1, \ldots, v_{r-1})\,.$$

HW 4.11. **(a)** Prove that $i_v : \bigwedge^r V \to \bigwedge^{r-1} V$ is a linear map.

(b) Given $\varphi \in \bigwedge^r V$ and $\psi \in \bigwedge^s V$, prove that

$$i_v(\varphi \wedge \psi) \;=\; (i_v \varphi) \wedge \psi \;+\; (-1)^r \, \varphi \wedge (i_v \psi)\,.$$

(c) Prove that $i_v(i_w\varphi) = -i_w(i_v\varphi)$.

HW 4.12. Let $(x, y, t) \in \mathbb{R}^3$. Let (u, v) be coordinates on $\mathbb{R}^2$ and define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $F(x, y, t) = (x\cos t, y \sin t)$.

(a) Set $X = t\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + xy\frac{\partial}{\partial t}$ and $p = (1, -1, 2) \in \mathbb{R}^3$. Compute F_*X_p.

(b) Write $F = (F^1, F^2)$. Compute $\mathrm{d}F^1 \wedge \mathrm{d}F^2$ and $i_X(\mathrm{d}F^1 \wedge \mathrm{d}F^2)$.

(c) Let $\alpha = \sqrt{u^2 + v^2 + 1}\,\mathrm{d}u \wedge \mathrm{d}v$. Compute $F^*(\alpha)$; and $F^*(\alpha_q)$, $q = F(p)$.

HW 4.13. Let $F : M \to N$ be a smooth map between manifolds. Suppose that $\alpha \in \Omega^k(N)$ and $\beta \in \Omega^\ell(N)$. Prove the following.

(a) $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$.

(b) If $\mathrm{d}\alpha = 0$ then $\mathrm{d}(F^*\alpha) = 0$.

HW 4.14. Let M be a manifold and V a vector space. Let $\alpha \in \Omega^k(M, V)$ be a V–valued k-form on M. If we chose a basis $\mathcal{B} = \{v_1, \ldots, v_r\}$ of V, then we can write $\alpha = \alpha^i\, v_i$, where $\alpha_i \in \Omega^k(M)$. Define

$$\mathrm{d}\alpha \;=\; (\mathrm{d}\alpha^i)\, v_i\,.$$

Prove that the definition of $\mathrm{d}\alpha$ is independent of our choice of basis $\mathcal{B}$.

Definition. Given an vector field V on M, let $\varphi_V(t, x)$ denote the *flow of* V through $x \in M$:

$$\varphi_V(0, x) \;=\; x \qquad \text{and } \frac{\partial}{\partial t}\,\varphi(t, x) \;=\; V_{\varphi_V(t,x)}\,.$$

It is a fact that given any $x_o \in M$ there exists a neighborhood U of x_o in M and $\varepsilon > 0$ such that $\varphi_V : (-\varepsilon, \varepsilon) \times W \to M$ is defined. Let

$$\varphi_{V,t} : W \to M$$

denote the map $\varphi_{V,t}(x) = \varphi_V(t, x)$.

Definition. Given a vector field V on M and $\alpha \in \Omega^k(M)$, define the *Lie derivative* of α by V to be

$$(\mathcal{L}_V\alpha)_x \;=\; \frac{\partial}{\partial t}\,(\varphi_{V,t})^*\alpha_{\varphi_{V,t}(x)}\Big|_{t=0}\,.$$

where φ_V is the flow of V.

HW 4.15. **(a)** Let $f : M \to \mathbb{R}$ be a smooth function, and V a smooth vector field. Prove that

$$\mathcal{L}_V f \;=\; V(f)\,.$$

(b) Let $\alpha \in \Omega^k(M)$. Prove that

$$\mathrm{d}\,(\mathcal{L}_V\alpha) \;=\; \mathcal{L}_V\,\mathrm{d}\alpha\,.$$

(c) Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$. Prove that

$$\mathcal{L}_V(\alpha \wedge \beta) \;=\; (\mathcal{L}_V\alpha) \wedge \beta \;+\; \alpha \wedge (\mathcal{L}_V\beta)\,.$$

HW 4.16. Let $\alpha \in \Omega^k(M)$. In this exercise you will prove that

$$\mathcal{L}_V\alpha \; = \; i_V \mathrm{d}\alpha \; + \; \mathrm{d}\, i_V\alpha \,.$$

To do this define $L_V\alpha := i_V\mathrm{d}\alpha \; + \; \mathrm{d}\, i_V\alpha$.

(a) Given $f \in \Omega^0(M) = \mathcal{C}^\infty(M)$, prove that $L_V f = Xf$.

(b) Prove that $d(L_V\alpha) = L_V\, \mathrm{d}\alpha$.

(c) Given $\beta \in \Omega^\ell(M)$, prove that $L_V(\alpha \wedge \beta) = (L_V\alpha) \wedge \beta + \alpha \wedge (L_V\alpha)$.

(d) Use Exercise 4.15 to conclude that $L_V\alpha = \mathcal{L}_V\alpha$.

5. Frobenius Theorem

Let M be an n-dimensional manifold. Let $\pi : TM \to M$ denote the tangent bundle.

Definition. A *rank m distribution* on M is a submanifold $\mathcal{D} \subset TM$ with the property that $\mathcal{D}_p = \mathcal{D} \cap \pi^{-1}(p)$ is a m–dimensional linear subspace of T_pM. The distribution is *smooth* if for all $p \in M$ there exist smooth vector fields $X_1, \ldots, X_m$ defined on a neighborhood U of p such that $\{X_1(q), \ldots, X_m(q)\}$ span $\mathcal{D}_q$ fo all $q \in U$. We refer to the $X_1, \ldots, X_m$ as a *local framing* of $\mathcal{D}$.

Assume. We assume that all distributions are smooth.

Definition. The distribution $\mathcal{D}$ is *integrable* if given any $p \in M$ there exists an embedded submanifold $P \subset M$ such that $T_qP = \mathcal{D}_q$ for all $q \in P$. Equivalently, $\mathcal{D}$ is integrable if every $p \in M$ admits coordinates $x : U \to \mathbb{R}^n$ such that $\mathrm{span}_{\mathcal{C}^\infty(U)}\{X_1, \ldots, X_m\} = \mathrm{span}_{\mathcal{C}^\infty(U)}\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$. We take $P := \{q \in U \mid x^s(q) = x^s(p)\}$.

Definition. The *annihilator of* $\mathcal{D}$ is $\mathrm{Ann}(\mathcal{D}) := \{(p, \mu) \in T^*M \mid \mu(v) = 0 \; \forall \; v \in \mathcal{D}_p\}$. Given $p \in M$ there always exists a neighborhood U and 1-forms $\eta^{m+1}, \ldots, \eta^n$ on U such that $\{\eta^{m+1}(q), \ldots, \eta^n(q)\}$ span $\mathrm{Ann}(\mathcal{D}_q)$ for all $q \in U$. We refer to $\eta^{m+1}, \ldots, \eta^n$ as a *local framing* of $\mathrm{Ann}(\mathcal{D})$.

Fix indices

$$1 \; \le \; a, b \; \le \; m \quad \text{ and } \quad m+1 \; \le \; r, s \; \le \; n \,,$$

and note that

$$\eta^s(X_a) = 0 \,.$$

Definition. The distribution $\mathcal{D}$ is *Frobenius* if 'sections of $\mathcal{D}$ are closed under the Lie bracket.' That is, if X, Y are two smooth vector fields defined on an open set $U \subset M$ and $X(q), Y(q) \in \mathcal{D}_q$ for all $q \in U$, then $[X, Y](q) \in \mathcal{D}_q$ as well.

Definition. The annihilator $\mathrm{Ann}(\mathcal{D})$ is *Frobenius* if given any local frame $\{\eta^{m+1}, \ldots, \eta^n\}$

$$\mathrm{d}\eta^s \; \equiv \; 0 \mod \; \eta^{m+1}, \ldots, \eta^n \,.$$

That is, there exist 1-forms β^s_r such that $\mathrm{d}\eta^s = \beta^s_r \wedge \eta^r$. (If the $\{\eta^s\}$ are defined on $U \subset M$, then the ideal in $\Omega(U)$ generated by the η^s is closed under exterior differentiation.)

5.1. **Lemmas.**

HW 5.1. Prove that $\mathcal{D}$ is Frobenius if and only if there exists some local framing $\{X_a\}$ of $\mathcal{D}$ on U such that $[X_a, X_b](q) \in \mathcal{D}_q$ for all $q \in U$.

HW 5.2. Prove that the definition of Frobenius for $\mathrm{Ann}(\mathcal{D})$ does not depend on our choice of local framing $\{\eta^s\}$.

HW 5.3. Prove that $\mathcal{D}$ is Frobenius if and only if $\mathrm{Ann}(\mathcal{D})$ is Frobenius.

HW 5.4. Prove that $\mathrm{Ann}(\mathcal{D})$ is always Frobenius when $m = 1$.

HW 5.5. Suppose that there exist local coordinates $x : U \subset M \to \mathbb{R}^n$ such that

$$\eta^{m+1}, \ldots, \eta^n \ \in \ \mathrm{span}_{\mathcal{C}^\infty(U)}\{\mathrm{d}x^{m+1}, \ldots, \mathrm{d}x^n\}\,.$$

Prove that $\mathrm{d}\eta^s \equiv 0 \mod \eta^{m+1}, \ldots, \eta^n$. That is, $\mathrm{Ann}(\mathcal{D})$ is Frobenius. (Note that this proves the 'easy' direction of the Frobenius Theorem.)

HW 5.6 (Vector fields)**.** Let $M = \mathbb{R}^3 - \{x = 0\}$ with standard Cartesian coordiantes (x, y, z). Prove that the vector fields

$$U \ = \ 5y\,\frac{\partial}{\partial x} \ + \ x\,\frac{\partial}{\partial y}\,, \qquad V \ = \ 3z\,\frac{\partial}{\partial x} \ - \ x\,\frac{\partial}{\partial z}$$

are pointwise linearly independent. Is the rank 2 distribution determined by $\mathcal{D} = \mathrm{span}\{U, V\}$ integrable? (That is, given $p \in M$ does there exist a 2-dimensional submanifold $P \subset M$ through p such that $T_qP = \mathcal{D}_q$ for all $q \in P$?) Find a no-where zero 1-form ω spanning $\mathrm{Ann}(\mathcal{D})$.

HW 5.7 (Forms)**.** Let $M = \mathbb{R}^3 - \{0\}$ with standard Cartesian coordinates (x, y, z). Set $\omega = z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z$. Is $\mathrm{span}\{\omega\}$ Frobenius? Given $p \in M$ does there exist a 2-dimensional submanifold $P \subset M$ through p such that $\omega \equiv 0$ on P?

6. LIE GROUPS

Notation. Let V be an n–dimensional vector space. Let $\mathrm{GL}(V) = \mathrm{Aut}(V)$ denote the set of invertible linear transformations $V \to V$. Let $\mathfrak{gl}(V) = \mathrm{End}(V)$ denote the set of all (not necessarily invertible) linear transformations $V \to V$.

Let $\mathfrak{g} = T_{\mathrm{Id}}G$ denote the Lie algebra of a Lie group G.

Definition. Fix $0 \neq \nu_o \in \bigwedge^n V^*$. The special linear group is

$$\mathrm{SL}(V, \nu_o) \ := \ \{A \in \mathrm{GL}(V) \mid A^*\nu_o = \nu_o\}\,.$$

HW 6.1. *(a)* Prove that the definition of $\mathrm{SL}(V, \nu_o)$ does not depend on the choice of $0 \neq \nu_o \in \bigwedge^n V^*$. That is, $\mathrm{SL}(V, \nu_o) = \mathrm{SL}(V)$.

(b) Prove that $\mathrm{SL}(V)$ is a Lie group of dimension $n^2 - 1$.

(c) Prove that the Lie algebra of $\mathrm{SL}(V)$ is

$$\mathfrak{sl}(V) \ = \ \{X \in \mathrm{End}(V) \mid \mathrm{trace}(X) = 0\}\,.$$

Definition. Let V be an n–dimensional vector space. Let $\omega_o \in \bigwedge^2 V^*$ be a skew-symmetric bilinear form. Then ω_o is *nondegenerate* if $v \lrcorner \omega_o \neq 0$ for all nonzero $v \in V$. Equivalently, given $0 \neq v \in V$, there exists $w \in V$ such that $\omega_o(v, w) \neq 0$.

HW 6.2. Prove that ω_o is nondegenerate if and only if there exists a basis $\{\alpha^1, \ldots, \alpha^{2m}\}$ of V^* such that $\omega_o = \alpha^1 \wedge \alpha^{m+1} + \cdots + \alpha^m \wedge \alpha^{2m}$. In particular, $n = 2m$ is even.

HW 6.3. Given a nondegenerate $\omega_o \in \bigwedge^2 V^*$ define

$$\mathrm{Sp}(n) \ = \ \mathrm{Sp}(V, \omega_o) \ := \ \{A \in \mathrm{GL}(V) \mid \omega_o(Av, Aw) = \omega_o(v, w)\,, \ \forall\, v, w \in V\}\,.$$

(a) Prove that $\mathrm{Sp}(n)$ is a Lie group. What is the dimension of $\mathrm{Sp}(n)$?

(b) Determine the Lie algebra $\mathfrak{sp}(n)$ of $\mathrm{Sp}(n)$.

Definition. Let G be a Lie group. Given $g \in G$ define diffeomorphisms $L_g, R_g : G \to G$ by

$$L_g(z) \ := \ gz\,, \qquad R_g(z) \ := \ zg\,.$$

HW 6.4. Let G be a Lie group. Given a function $g : M \to G$, prove that $\mathrm{d}g^{-1} = -(L_{g^{-1}})_*(R_{g^{-1}})_*\,\mathrm{d}g$.

Remark. In the case that $G \subset \mathrm{GL}_n R$, we may view g as a matrix-valued function on M. Then using the notation of matrix multiplication, we have $\mathrm{d}(g^{-1}) = -g^{-1}\,\mathrm{d}g\,g^{-1}$.

Definition. Let $\mathfrak{g} = T_{\mathrm{Id}}G$ be the Lie algebra of a Lie group G. Given $X \in \mathfrak{g}$, define a left-invariant vector field $\widehat{X} \in \mathfrak{X}(G)$ on G by

$$\widehat{X}_g \ := \ (L_g)_* X\,.$$

Let $\gamma_X(t) = \exp(tX)$ be the *flow of* $\widehat{X}$ *through* Id: that is, $\gamma_X(0) = \mathrm{Id}$ and $\dot\gamma_X(t) = \widehat{X}_{\gamma_X(t)}$.

HW 6.5. **(a)** Prove that $\exp(s+t)X = exp(sX)exp(tX)$.

(b) Prove that $\exp(tX)^{-1} = \exp(-tX)$.

Hint. It suffices to show that $\gamma(t) = \exp(tX)^{-1}$ is the flow of $-\widehat{X}$ through $\mathrm{Id} \in G$. To that end you may find $\mathrm{d}g^{-1} = g^{-1}\,\mathrm{d}g\,g^{-1}$ useful. (You will need to prove this identity before using it, if you have not yet seen it in class or proven it in a previous homework.)

Definition. Let G be a Lie group and $g \in G$. Define a diffeomorphism $C_g : G \to G$ by $C_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$. That is,

$$C_g(a) \ := \ gag^{-1}\,.$$

The differential of C_g is an invertible linear map $(C_g)_* : T_aG \to T_{C_g(a)}G$. Note that

$$(C_g)_* \ = \ (L_g)_*(R_{g^{-1}})_* \ = \ (R_{g^{-1}})_*(L_g)_*\,.$$

At $a = \mathrm{Id} \in G$, $(C_g)_*$ maps $\mathfrak{g} = T_{\mathrm{Id}}G$ to itself. This yields the *adjoint map* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, sending g to $\mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g})$, defined by

$$\mathrm{Ad}_g(X) \ := \ (C_g)_* X\,, \quad \forall\, X \in \mathfrak{g}\,.$$

HW 6.6. Prove that $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a group homomorphism: $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \circ \mathrm{Ad}_h$.

HW 6.7. Fix $g \in G$.

(a) Prove that $(C_g)_*\widehat{X} = \widehat{\mathrm{Ad}_g X}$. Equivalently, $(C_g)_*(\widehat{X}_{g^{-1}ag}) = (\widehat{\mathrm{Ad}_g X})_a$ for all $a \in G$.

(b) Prove that $\exp(t\mathrm{Ad}_g X) = C_g(\exp(tX))$.

(c) Prove that $(R_{g^{-1}})_* \widehat{X} = \widehat{\mathrm{Ad}_g X}$. Equivalently, $(R_{g^{-1}})_*(\widehat{X}_{ag}) = (\widehat{\mathrm{Ad}_g X})_a$ for all $a \in G$.

Definition. The *Maurer-Cartan* form θ is the $\mathfrak{g}$–valued 1-form on G defined by

$$\theta_g(v) \ := \ (L_{g^{-1}})_* v \,,$$

where $v \in T_g G$.

Remark. The Maurer-Cartan form θ is the unique left-invariant (${L_g}^*\theta = \theta$), $\mathfrak{g}$–valued 1-form with the property that ϑ is the identity on $\mathfrak{g} = T_{\mathrm{Id}} G$.

HW 6.8. Given $X \in \mathfrak{g}$, prove that $\theta(\widehat{X}) = X$.

HW 6.9. Let $\theta \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan form on G. Prove that ${R_g}^*\theta = \mathrm{Ad}_g(\theta)$. That is, given $v \in T_a G$, show that

$$({R_g}^* \theta_{ag}) v \ = \ \mathrm{Ad}_{g^{-1}}(\theta_a(v)) \,.$$

6.1. **Bi-invariant metrics.**

HW 6.10. Let G be a Lie group with bi-invariant metric $g = \langle \cdot , \cdot \rangle$. We have seen that $\langle [Z,X], Y\rangle_e + \langle X, [Z,Y]\rangle_e = 0$, for all $X, Y, Z \in T_e G = \mathfrak{g}$. Fix an orthonormal basis $\{X_1, \ldots, X_n\}$ of $T_e G = \mathfrak{g}$. Let $\{\widehat{X}_1, \ldots, \widehat{X}_n\} \subset \mathfrak{X}(G)$ denote the left-invariant, orthonormal framing of G associated to the basis. Define a connection on G by

$$\nabla_{\widehat{X}_i} \widehat{X}_j \ := \ \tfrac{1}{2} [\widehat{X}_i, \widehat{X}_j] \,.$$

Prove that ∇ is the Levi-Civita connection on $(G, \langle \cdot , \cdot \rangle)$. That is, prove that ∇ is torsion-free

$$\nabla_V W \ - \ \nabla_W V \ = \ [V, W] \qquad \forall \ V, W \in \mathfrak{X}(G)$$

and metric–compatible

$$\mathrm{d}\langle V, W\rangle \ = \ \langle \nabla V, W\rangle \ + \ \langle V, \nabla W\rangle \qquad \forall \ V, W \in \mathfrak{X}(G) \,.$$

6.2. **Homogeneous manifolds.**

Definition. A smooth *left-action* of a Lie group G on a manifold M is a smooth map $\phi : G \times M \to M$ such that

$$\begin{aligned} \phi(e, x) \ &= \ x \,, \quad \forall \ x \in M \,, \\ \phi(a \,,\, \phi(b, x)) \ &= \ \phi(ab, x) \,. \end{aligned}$$

Notation. We will often write $\phi(a, x)$ as ax. Let $\phi_a : M \to M$ denote the smooth map$\phi_a(x) = \phi(a, x)$.

Definition. The action is *transitive* if there exists $x_o \in M$ such that $\phi(G, x_o) = G\, x_o = M$.

HW 6.11. Prove that the action is transitive if and only if $G\, x = M$ for any $x \in M$.

Definition. A manifold M is *homogeneous for a Lie group* G if there exists a smooth, transitive left-action $\phi : G \times M \to M$ of G on M.

Example. *(a)* $M = S^n$ is homogeneous for $G = \mathrm{O}(n+1)$.

(b) $M = V \backslash \{0\}$ is homogeneous for $G = \mathrm{GL}(V)$.

(c) $M = \mathbb{P}V$ is homogeneous for $G = \mathrm{SL}(V)$.

Definition. Let (M, G) be a homogeneous space. The *isotropy subgroup* of $x \in M$ is $H_x := \{a \in G \mid ax\}$. (This is the stabilizer of x.)

HW 6.12. **(a)** Prove that H_x is a subgroup. (Closed under multiplication and inverses.)

(b) Prove that H_x is a closed subset of G.

(c) Given $x, y \in M$, prove that H_x and H_y are conjugate.

Definition. Let G be a Lie group and $H \subset G$ a subgroup. Define a left-action $\lambda : G \times G/H \to G/H$ on the coset space G/H by $\lambda(a, bH) = abH$. Let $\lambda_a : G/H \to G/H$ denote the map $\lambda_a(bH) = abH$.

HW 6.13. Prove that λ is transitive.

HW 6.14. Let G be a Lie group with Lie subgroup H, and let $\mathfrak{h} \subset \mathfrak{g}$ denote their Lie algebras. Define a distribution $\Delta \subset TG$ by $\Delta_a := (L_a)_* \mathfrak{h} \subset T_a G$.

(a) Prove that Δ is left-invariant: $(L_a)_* \Delta_b = \Delta_{ab}$.

(b) Let $a \in G$. Given $z \in aH \subset G$, prove that $T_z(aH) = \Delta_z \subset T_z G$.

7. Vector bundles

Definition. A *vector bundle E of rank* r over a manifold M is a smooth manifold with submersion $\pi : E \to M$ such that

(a) $E_x = \pi^{-1}(x)$ is a vector space of dimension r for all $x \in M$,

(b) there exists a covering $\{U_\alpha \mid \alpha \in A\}$ of M by open sets and diffeomorphisms φ_α such that the diagram below commutes,

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^r \arrow[dl, "proj_1"] \\
& U_\alpha &
\end{tikzcd}$$

(c) the restriction of φ_α to fibres is a linear isomorphism $E_x \to \mathbb{R}^r$ for all $x \in U_\alpha$.

Definition. The φ_α are *local trivializations* of E. It follows from the definition that the map $\varphi_\beta \circ \varphi_\alpha^{-1}$,

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^r \xrightarrow{\varphi_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\beta} (U_\alpha \cap U_\beta) \times \mathbb{R}^r\,,$$

is of the form

$$(x, v) \mapsto (x, g_{\beta\alpha}(x)\, v)$$

for some smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}_r\mathbb{R}$. The $g_{\beta\alpha}$ are the *transition functions* associated to the local trivializations.

Definition. Let E be a vector bundle over M. Define the *dual bundle* E^* to be the bundle with fibre $(E^*)_x = {E_x}^*$:

$$E^* \ := \ \{\lambda \in {E_x}^* \mid x \in M\}$$

with projection map $\pi^* : \lambda \in {E_x}^* \;\mapsto\; x \in M$.

HW 7.1. Assume E is a vector bundle of rank r. Prove E^* is a vector bundle of rank r.

Definition. Let E and F be vector bundles over M. Define the *product bundle* $E \otimes F$ to be the bundle with fibre $(E \otimes F)_x = E_x \otimes F_x$:

$$E \otimes F \;:=\; \{\xi \in E_x \otimes F_x \mid x \in M\}$$

with projection map $\pi : \xi \in E_x \otimes F_x \;\mapsto\; x \in M$.

HW 7.2. Set $\mathsf{rank}(E) = r$ and $\mathsf{rank}(F) = s$. Prove $E \otimes F$ is a vector bundle of rank rs.

Definition. Let E be a vector bundle over M. Define $\bigwedge^2 E$ to be the vector bundle over M with fibre $(\bigwedge^2 E)_x = \bigwedge^2(E_x)$:

$$\textstyle\bigwedge^2 E \;:=\; \{\xi \in \bigwedge^2 E_x \mid x \in M\}\,.$$

HW 7.3. Assume E is a vector bundle of rank r. Prove that $\bigwedge^2 E$ is a vector bundle of rank $\frac{1}{2}r(r-1)$.

Definition. Given an n-dimensional vector space V consider the Grassmannian $\mathrm{Gr}(k,V)$ of k-planes in V. The *tautological vector bundle* $\pi : \mathcal{T} \to \mathrm{Gr}(k,V)$ over $\mathrm{Gr}(k,V)$ is a rank k vector bundle. The fibre over $\xi \in \mathrm{Gr}(k,V)$ is the k-dimensional vector space $\xi \subset V$. That is,

$$\mathcal{T} \;=\; \{(\xi, v) \in \mathrm{Gr}(k,V) \times V \mid v \in \xi\}\,,$$

and the projection $\pi : \mathcal{T} \to \mathrm{Gr}(k,V)$ maps $(\xi, v) \mapsto \xi$.

Example. Recall that $\mathrm{Gr}(1,V) = \mathbb{P}V$.

HW 7.4. Prove $\mathcal{T}$ is a vector bundle, and identify the transition functions.

8. EUCLIDEAN GEOMETRY

8.1. **Structure Equations.** Let $\langle \cdot, \cdot \rangle$ denote a positive definite inner product on $\mathbb{R}^n$.

Definition. Let $\mathrm{O}(n) = \{e = (e_1, \ldots, e_n) \mid e \text{ an orthonormal basis of } \mathbb{R}^n\ \}$.

Remark. If we fix a basis of $\mathbb{R}^n$, then we may identify $\mathrm{O}(n)$ with the set of $n \times n$ orthogonal matrices. (View e_i as a column vector.) To be precise: fix $b = (b_1, \ldots, b_n) \in \mathrm{O}(n)$, then given any other orthonormal frame e of $\mathbb{R}^n$ there exists a unique orthogonal matrix $A = A(e)$ such that $e = Ab$.

Definition 8.1. Set

$$\mathrm{Euc}(n) \;:=\; \left\{ \left(\begin{array}{cc} 1 & 0 \\ x & e \end{array} \right) \;\middle|\; x \in \mathbb{R}^n\,,\ e \in \mathrm{O}(n) \right\} \;\simeq\; \mathbb{R}^n \times \mathrm{O}(n)\,.$$

Define $\omega^i, \theta^i_j \in \Omega^1_{\mathrm{Euc}(n)}$ by

$$\mathrm{d}x \;=\; \omega^i\, e_i \quad \text{and} \quad \mathrm{d}e_i \;=\; \theta^j_i\, e_j\,.$$

HW 8.1. Prove that $\theta^i_j + \theta^j_i = 0$.

Definition. Let $\mathfrak{so}(n)$ denote the vector space of skew-symmetric $n \times n$ matrices.

Remark. We say that θ is $\mathfrak{so}(n)$–valued.

HW 8.2. Prove that $\mathfrak{so}(n)$ is a Lie algebra.

HW 8.3. Prove that the $\{\omega^i, \theta^i_j \mid i < j\}$ form a co-framing on $\mathrm{Euc}(n)$.

HW 8.4. Let

$$\vartheta \;=\; \begin{pmatrix} 0 & 0 \\ \omega & \theta \end{pmatrix} .$$

Prove that $\mathrm{d}\vartheta = -\vartheta \wedge \vartheta$.

HW 8.5. Let

$$a \;=\; \begin{pmatrix} 1 & 0 \\ x & e \end{pmatrix} , \quad b \;=\; \begin{pmatrix} 1 & 0 \\ y & f \end{pmatrix} \;\in\; \mathrm{Euc}(n) .$$

Define a product on $\mathrm{Euc}(n)$ by matrix multiplication:

$$a \cdot b \;=\; \begin{pmatrix} 1 & 0 \\ x + ey & ef \end{pmatrix} .$$

Prove that $\mathrm{Euc}(n)$ is a Lie group with this product.

HW 8.6. In this exercise you will prove that

$$\vartheta \;=\; \begin{pmatrix} 0 & 0 \\ \omega & \theta \end{pmatrix}$$

is a left-invariant 1-form on the Lie group $\mathrm{Euc}(n)$.

(a) Let

$$A = \begin{pmatrix} 0 & 0 \\ X & E \end{pmatrix} \;\in\; T_a\mathrm{Euc}(n) .$$

Prove that $\mathrm{d}x_{|a}(A) = X \in \mathbb{R}^3$ and $\mathrm{d}e_{i|a}(A) = E_i$ (where E_i is the i-th column of E).

(b) Prove that

$$(L_b)_* A \;=\; a \cdot A \;=\; \begin{pmatrix} 0 & 0 \\ fX & fE \end{pmatrix} \in T_{ba}\mathrm{Euc}(n) .$$

(c) Use (a) and (b) to prove that

$$(L_b^*\, \mathrm{d}x_{|ba})(A) \;=\; fX \quad \text{and} \quad (L_b^*\, \mathrm{d}e_{i|ba})(A) \;=\; (fE)_i .$$

(d) Use (c) to prove that

$$(L_b^*\, \omega^i_{|ba})(A) \;=\; \omega^i_{|a}(A) \quad \text{and} \quad (L_b^*\, \theta^i_{j|ba})(A) \;=\; \theta^i_{j|a}(A)$$

for all $A \in T_a\mathrm{Euc}(n)$. That is, ϑ is left-invariant.

8.2. **Framings of $\mathbb{R}^n$.** Let $e = (e_1, \ldots, e_n) : \mathbb{R}^n \to \mathrm{O}(n)$ be an orthonormal frame on $\mathbb{R}^n$. Suppose that $\tilde{e}$ is a second orthonormal framing. Then there exists a matrix-valued $A : \mathbb{R}^n \to \mathrm{O}(n)$ such that $\tilde{e} = Ae$. That is, if we write $A = (A^i_j)$, $i, j = 1, \ldots, n$, then

$$\tilde{e}_j = A^i_j \, e_i \, . \tag{8.2}$$

HW 8.7 (Associated coframes)**.** Let $\{\omega^i\}_{i=1}^n$ and $\{\tilde{\omega}^i\}_{i=1}^n$ respectively denote the dual coframings. In analogy with (8.2) express $\tilde{\omega}^j$ in terms of A and ω.

HW 8.8 (Associated connection forms)**.** Recall that the connection forms $\theta = (\theta^i_j)$ (with respect to the framing e) are defined by $\mathrm{d}e_j = \theta^i_j \, e_i$. Similarly the connection forms $\tilde{\theta} = (\tilde{\theta}^i_j)$ (with respect to the framing $\tilde{e}$ are defined by $\mathrm{d}\tilde{e}_j = \tilde{\theta}^i_j \, \tilde{e}_i$. As above, express $\tilde{\theta}^i_j$ in terms of θ and A.

8.3. **Submanifolds $M \subset \mathbb{R}^n$.** Suppose that M is an m-dimensional submanifold of $\mathbb{R}^n$. Fix index ranges

$$1 \le i, j \le n \, , \quad 1 \le a, b \le m \, , \quad m < s, t \le n \, .$$

Assume throughout that $e : U \to \mathrm{O}(n)$ is a local, first-order adapted moving frame of $\mathbb{R}^n$ defined on an open $U \subset M$, and ω the associated coframing.

HW 8.9 (Change of frame)**.** Let $\tilde{e}$ be a second local, first-order adapted moving frame of $\mathbb{R}^n$ on U. As in §8.2 there exists $A : U \to \mathrm{O}(n)$ such that $\tilde{e} = A\,e$. Prove that A takes values in $\mathrm{O}(m) \times \mathrm{O}(n-m)$. That is,

$$A \;=\; \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

with $B \in \mathrm{O}(m)$ and $C \in \mathrm{O}(n-m)$.

HW 8.10 (First fundamental form)**.** Let $\mathbf{I} = \sum_a \omega^a \circ \omega^a$ denote the First Fundamental form on $U \subset M$. Prove that $\mathbf{I}$ does not depend on our choice of framing e. That is, if $\tilde{e}$ is a second local, first-order adapted moving frame of $\mathbb{R}^n$ on U, and $\tilde{\omega}$ is the associated coframe, then $\sum_a \omega^a \circ \omega^a = \sum_a \tilde{\omega}^a \circ \tilde{\omega}^a$.

HW 8.11 (Second fundamental form)**.** Recall that the coefficients h^s_{ab} of the second fundamental form (with respect to the framing e) are defined by $\theta^s_a = h^s_{ab}\,\omega^b$. Let $\tilde{e}$ be a second local first-order adapted framing. The coefficients $\tilde{h}^s_{ab}$ of the second fundamental form (with respect to the framing $\tilde{e}$ are defined by $\tilde{\theta}^s_a = \tilde{h}^s_{ab}\tilde{\omega}^b$.

(a) Express $\tilde{h}^s_{ab}$ in terms of h and A.

(b) Prove that the definition of **II** is independent of our choice of framing. That is, $h^s_{ab}\, e_s \otimes \omega^a \circ \omega^b = \tilde{h}^s_{ab}\, \tilde{e}_s \otimes \tilde{\omega}^a \circ \tilde{\omega}^b$.

HW 8.12 (Connection form and Gauss curvature)**.** Let $M \subset \mathbb{R}^3$ be a surface, and let $e : U \subset M \to \mathrm{SO}(3)$ be a local adapted framing. Let $\{\omega^i\}$ denote the dual coframing and $\theta = (\theta^i_j)$ the connection 1-forms. Prove that

$$\mathrm{d}\theta^1_2 \;=\; K\,\omega^1 \wedge \omega^2 \, ,$$

where K is the Gauss curvature of M.

HW 8.13. The helicoid is a surface $S \subset \mathbb{R}^3$ parameterized by

$$(t,u) \;\mapsto\; (u\cos t, u\sin t, c\,t)\,,$$

$0 \neq c \in \mathbb{R}$. (The helicoid is *not* a surface of revolution.)

(a) Construct a local orthonormal moving frame of $\mathbb{R}^3$ on S.

(b) Compute the connection forms θ^i_j with respect to the framing.

(c) Compute the matrix $h = (h_{ab})$ of coefficients of the second fundamental form (with respect to the the moving frame).

(d) Compute the Gaussian and mean curvatures of the helicoid.

8.4. **Covariant differentiation.** Let ∇ denote covariant differentiation on $M^m \subset \mathbb{R}^n$: If $v \in T_pM$ and $X \in \mathfrak{X}(M)$ is a vector field on M, then

$$\nabla_v X \;:=\; \mathrm{proj}_{T_pM} D_v X \;\in\; T_pM\,.$$

HW 8.14. Let $X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. Prove that

(a) $\nabla_{fX} Y \;=\; f\,\nabla_X Y$,

(b) $\nabla_X fY \;=\; X(f)Y \;+\; f\,\nabla_X Y$,

(c) $\nabla_{X+Y} Z \;=\; \nabla_X Z \;+\; \nabla_Y Z$,

(d) $\nabla_X (Y+Z) \;=\; \nabla_X Y \;+\; \nabla_X Z$.

HW 8.15 (Metric compatible)**.** Given $u, v \in T_pM$, let $\langle u, v\rangle$ denote the inner product. Let $X, Y \in \mathfrak{X}(M)$ be vector fields on M. Prove that

$$\mathrm{d}\langle X, Y\rangle \;=\; \langle \nabla X, Y\rangle \;+\; \langle X, \nabla Y\rangle\,.$$

That is, given $v \in T_pM$,

$$v\,\langle X, Y\rangle \;=\; \mathrm{d}\langle X, Y\rangle(v) \;=\; \langle \nabla_v X, Y\rangle \;+\; \langle X, \nabla_v Y\rangle\,.$$

HW 8.16 (Torsion-free)**.** Let $X, Y \in \mathfrak{X}(M)$ be vector fields on M. Prove that

$$\nabla_X Y \;-\; \nabla_Y X \;=\; [X, Y]\,.$$

HW 8.17. Fix a local adapted framing e of $\mathbb{R}^n$ on M, and consider the associated dual 1-forms ω^a (cf. §8.7). Given $v \in T_pM$, we define the covariant derivative $\nabla_v \omega^a \in T^*_pM$ of ω^a by v by the forcing the Leibniz rule. That is, $\nabla_v \omega^a$ is defined by

$$0 \;=\; \mathrm{d}\,\delta^a_b\,(v) \;=\; \mathrm{d}\,(\omega^a(e_b))(v) \;=:\; (\nabla_v\,\omega^a)(e_b) \;+\; \omega^a(\nabla_v\, e_b)\,.$$

Prove that $\nabla_v\,\omega^a \;=\; -\theta^a_b(v)\,\omega^b$.

8.5. **Geodesics on surfaces of revolution.** Let $S \subset \mathbb{R}^3$ be a surface of revolution parameterized by

$$(s,t) \;\mapsto\; (f(s)\sin t, f(s)\cos t, g(s))\,,$$

with $f > 0$ and $\dot f^2 + \dot g^2 = 1$ (the curve $(0, f(s), g(s))$ has unit speed).

HW 8.18 (Meridians)**.** Prove that every meridian $t = t_o$ is a geodesic.

HW 8.19 (Parallels)**.** When is the parallel $s = s_o$ a geodesic?

8.6. **Curves on surfaces.** Let $M \subset \mathbb{R}^3$ be a surface. Suppose that $\gamma : (-\varepsilon, \varepsilon) \to M$ is a unit speed curve. We construct an orthonormal framing $(e_1(s), e_2(s), e_3(s))$ of $\mathbb{R}^3$ over γ as follows. Define $e_1(s) := \dot\gamma(s)$. Let $e_3(s) \in N_{\gamma(s)}M$ be a (smooth) choice of unit normal vector. (Such a choice is always possible for small s.) Finally, define $e_2(s) := e_3(s) \times e_1(s) \in T_{\gamma(s)}M$.

HW 8.20 (Structure equations)**.** Prove that there exist functions $k(s)$, $g(s)$ and $t(s)$ such that

$$\begin{aligned} \dot e_1 &= g\, e_2 + k\, e_3 \\ \dot e_2 &= -g\, e_1 + t\, e_3 \\ \dot e_3 &= -k\, e_1 - t\, e_2 \,. \end{aligned}$$

Definition. The curve γ is *principal* if $\dot\gamma(s)$ is an eigenvector of $\mathrm{d}e_3 : T_{\gamma(s)}M \to T_{\gamma(s)}M$. The curve is *asymptotic* if $\mathbf{II}(\dot\gamma(s), \dot\gamma(s)) \equiv 0$ (equivalently, $\dot\gamma \perp \mathrm{d}e_3(\dot\gamma)$). Prove that γ is

$$\begin{aligned} \text{geodesic} &\iff g = 0 \\ \text{asymptotic} &\iff k = 0 \\ \text{principal} &\iff t = 0 \,. \end{aligned}$$

The function $k(s)$ is the *normal curvature* of γ; the function $g(s)$ is the *geodesic curvature* of γ.

9. G–STRUCTURES

Notation. Let M be an n–dimensional manifold.

Definition. The *coframe bundle* over M is

$$\mathcal{F} := \{(x, u_x) \mid x \in M,\ u_x : T_xM \to \mathbb{R}^n \text{ is a linear isomorphism}\}\,.$$

Let $\pi : \mathcal{F} \to M$ denote the projection. The right-action of $\mathrm{GL}_n\mathbb{R}$ on $\mathcal{F}$ is defined by

$$u \cdot g := g^{-1} \circ u\,.$$

This action is free and transitive on the fibres $\pi^{-1}(x) =: \mathcal{F}_x$.

Definition. Given a Lie subgroup $G \subset \mathrm{GL}_n\mathbb{R}$, a *$G$–structure on M* is a sub-bundle $\mathcal{G} \subset \mathcal{F}$ with fibre group G.

Example. The frame bundle $\mathcal{F}$ is a $\mathrm{GL}_n\mathbb{R}$–structure on M.

Example. Let $\cdot$ be the dot product on $\mathbb{R}^n$ and g a Riemannian metric on M. Then

$$\mathcal{F}^g := \{(x, u_x) \mid x \in M,\ u_x : (T_xM, g_x) \to (\mathbb{R}^n, \cdot) \text{ is a linear isometry}\}$$

is an $\mathrm{O}(n)$–structure on M.

Definition. Let V be an n–dimensional vector space. Fix $0 \neq \nu_o \in \bigwedge^n V^*$. (For example, if $\{\mathsf{v}^1 \ldots, \mathsf{v}^n\}$ is a basis of V^*, then set $\nu_o = \mathsf{v}^1 \wedge \cdots \wedge \mathsf{v}^n$.) The special linear group is

$$\mathrm{SL}(V) := \{A \in \mathrm{GL}(V) \mid A^*\nu_o = \nu_o\}\,.$$

Remark. Given any $A \in \mathrm{End}(V)$, $A^*\nu_o = \det(A)\,\nu_o$.

HW 9.1. Prove that M admits an $\mathrm{SL}(V)$–structure if and only if M is orientable.

Definition. Let V be an n–dimensional vector space. Let $\omega_o \in \bigwedge^2 V^*$ be a skew-symmetric bilinear form. Then ω_o is *nondegenerate* if $v \lrcorner \omega_o \neq 0$ for all nonzero $v \in V$. Equivalently, given $0 \neq v \in V$, there exists $w \in V$ such that $\omega_o(v,w) \neq 0$.

Definition. Given a nondegenerate $\omega_o \in \bigwedge^2 V^*$ define

$$\mathrm{Sp}(n) \ = \ \mathrm{Sp}(V,\omega_o) \ := \ \{A \in \mathrm{GL}_n\mathbb{R} \mid \omega_o(Av,Aw) = \omega_o(v,w)\,,\ \forall\ v,w \in V\}\,.$$

Definition. A 2-form $\omega \in \Omega^2(M)$ is *nondegenerate* if $\omega_x \in \bigwedge^2 T^*_x M$ is nondegenerate for all $x \in M$.

HW 9.2. Prove that M has a nondegenerate 2-form ω if and only if M admits an $\mathrm{Sp}(n)$–structure.

Definition. Let $\pi : \mathcal{G} \to M$ be a G–structure, let $u \in \mathcal{G}$ and $x = \pi(u) \in M$. The *vertical subspace* at u is

$$V_u \ := \ \ker\{\pi_* : T_u\mathcal{G} \to T_xM\} \ = \ \{\xi \in T_u\mathcal{G} \mid \eta_u(\xi) = 0\}\,.$$

HW 9.3. Prove that ξ is vertical if and only if ξ is tangent to a fibre $\mathcal{F}_x$.

Definition. The *Solder form* $\eta \in \Omega^1(\mathcal{F},\mathbb{R}^n)$ is the $\mathbb{R}^n$–valued 1-form on $\mathcal{F}$ defined by

$$\eta_u(\xi) \ := \ u\left(\pi_x(\xi)\right)\,, \quad \xi \in T_u\mathcal{F}\,.$$

The Solder form is also known as the *canonical* $\mathbb{R}^n$*–valued semi-basic form.* Given a fixed basis $\{\mathsf{v}_1,\ldots,\mathsf{v}_n\}$ of $\mathbb{R}^n$, define $\eta^j \in \Omega^1(\mathcal{F})$ by $\eta(\xi) =: \eta^j(\xi)\mathsf{v}_j$.

HW 9.4. Prove that ξ is vertical if and only if $\eta(\xi) = 0$.

Definition. A k-form $\varphi \in \Omega^k(\mathcal{F})$ is *semi-basic* if $\xi \lrcorner \varphi = 0$ for all vertical $\xi \in T\mathcal{F}$.

HW 9.5. Prove that φ is semi-basic if and only if $\varphi = \varphi_I \eta^I$ for some functions $\varphi_I : \mathcal{F} \to \mathbb{R}$.

Definition. Assume that $G \subset \mathrm{GL}_n\mathbb{R}$ is a Lie subgroup, let $\pi : \mathcal{G} \to M$ be a G–structure and define a vector field $\widetilde{X} \in \mathfrak{X}(\mathcal{G})$ on $\mathcal{G}$ by

$$\widetilde{X}_u \ := \ \frac{\mathrm{d}}{\mathrm{d}t}\, u \cdot \exp(tX)\Big|_{t=0}\,.$$

HW 9.6. Let $\pi : \mathcal{G} \to M$ be a G–structure.

(a) Prove that $\widetilde{X}$ is tangent to the fibres $\mathcal{G}_x = \pi^{-1}(x)$.

(b) Prove that $\widetilde{X}$ is vertical.

HW 9.7. Let $\{X_1,\ldots,X_r\}$ be a basis of $\mathfrak{g}$. Prove that $\{\widetilde{X}_1,\ldots,\widetilde{X}_r\}$ span $T_u\mathcal{G}_x = V_u$.

Definition. Given $g \in G$, let $\mathcal{R}_g : \mathcal{G} \to \mathcal{G}$ denote the induced diffeomorphism

$$\mathcal{R}_g(u) \ := \ u \cdot g \ = \ g^{-1} \circ u\,.$$

Remark. Note that $\mathcal{R}_g$ preserves the fibres of $\mathcal{G}$: $\mathcal{R}_g(\mathcal{G}_x) = \mathcal{G}_x$.

Definition. A *connection form* on a G–structure $\mathcal{G}$ is a $\mathfrak{g}$–valued 1-form $\vartheta \in \Omega^1(\mathcal{G},\mathfrak{g})$ with the properties that

$$\vartheta(\widetilde{X}) \ = \ X \quad \text{and} \quad {\mathcal{R}_g}^*\vartheta \ = \ \mathrm{Ad}_{g^{-1}}\vartheta\,.$$

Remark. In gory detail, with the base point notation, the second property reads as follows. Given $\xi \in T_u\mathcal{G}$, we must have

$$(\mathcal{R}_g{}^*\vartheta_{u\cdot g})\xi \;=\; \mathrm{Ad}_{g^{-1}}\left(\vartheta_u(\xi)\right) .$$

HW 9.8 (Toy example)**.** Suppose that M is a two-dimensional, oriented Riemannian manifold. Let $\pi : \mathcal{G}^+ \to M$ denote the bundle of oriented, orthonormal frames on M. Given $x \in M$ observe that the fibre $\mathcal{G}^+_x$ is naturally identified with the unit circle $S^1 \subset T_xM$: elements of S^1 are unit vectors $e_1 \in T_xM$. There is a unique $e_2 \in T_xM$ with the property that $\{e_1, e_2\}$ is an oriented, orthonormal basis of T_xM.

More generally $\mathcal{G}^+$ is locally of the form $U \times S^1$. To see this, let $x_o \in M$. Then x_o admits a neighborhood U with a local, oriented, orthonormal framing $\mathsf{e} = \{\mathsf{e}_1, \mathsf{e}_2\}$. This framing is a local section $\mathsf{e} : U \to \mathcal{G}^+$. Given any $e = \{e_1, e_2\} \in \pi^{-1}(U)$ there exists a unique $\tau = \tau(e) = \exp(it) \in S^1$ such that $e_1 = \mathsf{e}_1 \cos t - \mathsf{e}_2 \sin t$ and $e_2 = \mathsf{e}_1 \sin t + \mathsf{e}_2 \cos t$. So we define a diffeomorphism

$$\Phi : \pi^{-1}(U) \;\to\; U \times S^1 \quad \text{by} \quad \Phi(e) := (\pi(e), \tau(e))\,.$$

(a) Let $(x^1, x^2) : U \to \mathbb{R}^2$ be local coordinates on U, and $t \to \exp(it)$ be local coordinates on S^1. View (x^1, x^2, t) as coordintes on $\pi^{-1}(U) \simeq U \times S^1$. Express the Solder forms η^1, η^2 in terms of (x^1, x^2, t).

(b) Show that there exists a unique $\theta \in \Omega^1(U \times S^1)$ such that $\mathrm{d}\eta^1 = -\theta \wedge \eta^2$ and $\mathrm{d}\eta^2 = \theta \wedge \eta^1$.

(c) Argue that θ is globally defined. That is, $\theta \in \Omega^1(\mathcal{G}^+)$.

(d) Prove that

$$\vartheta \;=\; \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \;\in\; \Omega^1(\mathcal{G}^+, \mathfrak{so}(2))$$

is a connection form.

HW 9.9. Let ϑ be a connection form on $\mathcal{G}$. Prove that ϑ and η span the 1-forms on $\mathcal{G}$.

Hint. It suffices to show that $\eta(\xi) = 0 = \vartheta(\xi)$ implies that $\xi \in T\mathcal{G}$ is the zero vector.

HW 9.10. Fix $g \in G$ and $X \in \mathfrak{g}$. Prove that $(\mathcal{R}_g)_*\widetilde{X} = \widetilde{\mathrm{Ad}_{g^{-1}}X}$. That is,

$$(\mathcal{R}_g)_*\widetilde{X}_u \;=\; (\widetilde{\mathrm{Ad}_{g^{-1}}X})_{u\cdot g} \quad \text{for any } u \in \mathcal{G}\,.$$

Definition. Let $\pi : \mathcal{G} \to M$ be a G–structure. Fix a connection $\vartheta \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Let $u \in \mathcal{G}$ and $x = \pi(u) \in M$. The *horizontal subspace at u (with respect to the connection ϑ)* is

$$H^\vartheta_u \;=\; H_u \;:=\; \{\xi \in T_u\mathcal{G} \mid \vartheta_u(\xi) = 0\}\,.$$

HW 9.11. *(a)* Prove that $T_u\mathcal{G} = V_u \oplus H_u$.

(b) Prove that $(\mathcal{R}_g)_*H_u = H_{u\cdot g}$.

10. Riemannian metrics

Assume. M is an n-dimensional Riemannian manifold.

Definition. A *(Riemannian) metric* g on M assigns to each point $z \in M$ an inner-product g_z on T_zM. In particular, given $u, v \in T_zM$, $g_z(u,v) = g_z(v,u)$ and $g_z(u,u) > 0$ whenever $u \neq 0$.

Definition. A metric g on M is *smooth* if, given any two smooth vector fields X, Y on M, the function $f(z) := g_z(X_z, Y_z)$ is smooth.

Remark. We will often drop the base point notation and write $g_z(u,v)$ as $g(u,v)$.

HW 10.1. Given local coordinates (U,x) on M we may define functions $g_{ij} : U \to \mathbb{R}$ by $g_{ij} = g(\partial_{x^i}, \partial_{x^j})$. Prove that g is smooth if any only if the g_{ij} are smooth functions on U for all local coordinates (U,x).

HW 10.2. Prove that (g_{ij}) is a positive definite, symmetric matrix.

Assume. From this point on, we assume that *all Riemannian metrics are smooth.*

10.1. Covariant differentiation.

Let M be an n–dimensional manifold, $p \in M$, $v \in T_pM$ and $X \in \mathfrak{X}(M)$.

Definition 10.1. *Covariant differentiation at* p is a pairing $T_pM \times \mathfrak{X}(M) \to T_pM$, which we denote by $(v, X) \mapsto \nabla_v X$, that satisfies the following properties:

(i) (Linearity 1) $\nabla_v(X+Y) = \nabla_v X + \nabla_v Y$.
(ii) (Linearity 2) $\nabla_{u+v} X = \nabla_u X + \nabla_v Y$.
(iii) (Leibniz rule) $\nabla_v fX = \mathrm{d}f(v)\, X + f \nabla_v X$.
(iv) (Smoothness) $\nabla_Y X \in \mathfrak{X}(M)$.

for all $X, Y \in \mathfrak{X}(M)$, $u, v \in T_pM$ and $f \in \mathcal{C}^\infty(M)$. A choice of covariant differentiation on M is also called a *connection* on M.

HW 10.3. Let $U \subset M$ be a neighborhood of $p \in M$ that admits a local framing $X_1, \ldots, X_n \in \mathfrak{X}(U)$. Given $q \in M$ and $v \in T_qM$ define $\theta^i_j(v)_q \in \mathbb{R}$ by

$$\nabla_v X_i \;=\; \theta^j_i(v)_q \, X_j \, .$$

Prove that $\theta^i_j \in \Omega^1(U)$.

Definition 10.2. The $\theta = (\theta^i_j)$ are the *local connection 1-forms* associated to a choice of framing $X_1, \ldots, X_n$.

HW 10.4. Consider the Riemannian manifold $(M, g) = (\mathbb{R}^\mathsf{N}, \cdot)$. Let $D_v X$ denote directional derivative on $\mathbb{R}^\mathsf{N}$.

(a) Prove that D is a torsion-free, metric compatible connection.
(b) Compute θ with respect to the global framing $\partial/\partial x^1, \ldots, \partial/\partial x^\mathsf{N}$.

HW 10.5. Suppose that $M^n \subset \mathbb{R}^\mathsf{N}$ be a smooth embedded manifold. Let g denote the metric on M that is induced by the inner product on $\mathbb{R}^\mathsf{N}$. Given $v \in T_pM$ and $X \in \mathfrak{X}(M)$, define

$$\nabla_v X \;:=\; \mathrm{proj}_{T_pM} D_v X \, .$$

Prove that ∇ is a torsion-free, metric compatible connection on M.

10.2. **Parallel Transport.**

Notation. Let (M, g) be a Riemannian manifold, and let ∇ denote the Levi-Civita connection on M.

Definition. Let $\gamma : [0,1] \to M$ be a curve and $x = \gamma(0)$. Given $X_o \in T_xM$, the *parallel transport of X_o along γ* is a vector field $X(t) \in T_{\gamma(t)}M$ along γ such that $X(0) = X_o$ and $\nabla_{\dot\gamma}X = 0$.

Remark. Note that the parallel transport $X(t)$ of X_o is the solution of a first-order ordinary differential equation with initial value. In particular, solutions exist (at least for small t) and are unique.

HW 10.6. Let M be the 2-dimensional cylinder $\{x = (x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 = 1\}$. Fix the point $x_o = (1, 0, 0)$, and $X_o = \frac{\partial}{\partial x^2} \in T_{x_o}M$.

(a) Compute the parallel transport of v along the curve

$$\gamma(t) \ = \ (\cos(t), \sin(t), 0)\,.$$

(b) Compute the parallel transport of v along the curve

$$\gamma(t) \ = \ (\cos(t), \sin(t), t)\,.$$

Remark. As a surface in $\mathbb{R}^3$, M inherits a Riemannian metric g from the inner product on $\mathbb{R}^3$. Recall that, given $X \in \mathcal{X}(M)$ and $v \in T_xM$,

$$\nabla_v X \ = \ \mathrm{proj}_{T_xM} D_v X$$

where D_vX is the directional derivative of the vector field X, and proj_{T_xM} denote the orthogonal projection $\mathbb{R}^3 \to T_xM$.

HW 10.7. (a) Let M be the 2–sphere $\{x = (x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$. Let $x_o = (\sqrt{1/2}, 0, \sqrt{1/2})$. Compute the parallel transport $X(t)$ of $X_o = \frac{\partial}{\partial x^2} \in T_{x_o}M$ along

$$\gamma(t) \ = \ (\sqrt{1/2}\cos(t), \sqrt{1/2}\sin(t), \sqrt{1/2})\,.$$

(b) Note that $x_o = \gamma(0) = \gamma(2\pi)$. Compute $X(2\pi)$.

10.3. **Curvature.** Let (M, g) be a Riemannian manifold, and $\mathcal{F} \otimes M$ the $\mathrm{O}(n)$–bundle of orthonormal coframes on M. Let $\eta \in \Omega^1(\mathcal{F}, \mathbb{R}^n)$, $\vartheta \in \Omega^1(\mathcal{F}, \mathfrak{o}(n))$ and $\Omega = \mathrm{d}\vartheta + \vartheta \wedge \vartheta \in \Omega^2(\mathcal{F}, \mathfrak{o}(n))$ be the Solder 1-form, the Levi-Civita connection 1-form and the curvature 2-form, respectively. Recall that

$$\Omega \ = \ \mathrm{d}\vartheta + \vartheta \wedge \vartheta \ = \tfrac{1}{2}R(\eta \wedge \eta)\,,$$

with $R : \mathcal{F} \to \mathfrak{o}(n) \otimes \bigwedge^2(\mathbb{R}^n)^*$.

Recall that every $u \in \mathcal{F}$ is a isometry $u : (T_xM, g_x) \to (\mathbb{R}^n, \cdot)$, $x = \pi(u)$. Fix an orthonormal basis $\{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ of $\mathbb{R}^n$ with dual basis $\{\mathsf{v}^1, \ldots, \mathsf{v}^n\}$. Define $\eta^i, \vartheta^i_j = -\vartheta^j_i \in \Omega^1(\mathcal{F})$ and $\Omega^i_j \in \Omega^2(\mathcal{F})$ by

$$\eta \ = \ \eta^i\,\mathsf{v}_i\,, \quad \vartheta \ = \ \vartheta^i_j\,\mathsf{v}_i \otimes \mathsf{v}^j\,, \quad \Omega \ = \ \Omega^i_j\,\mathsf{v}_i \otimes \mathsf{v}^j\,.$$

Then $\Omega^i_j = \mathrm{d}\vartheta^i_j + \vartheta^i_k \wedge \vartheta^k_j$ and $R = R^i_{jk\ell}(\mathsf{v}_i \otimes \mathsf{v}^j) \otimes (\mathsf{v}^k \wedge \mathsf{v}^\ell)$, $R^i_{jk\ell} : \mathcal{F}^g \to \mathbb{R}$, and we have

$$\begin{aligned} \mathrm{d}\vartheta^i_j \ + \ \vartheta^i_k \wedge \vartheta^k_j \quad &= \quad R^i_{jk\ell}\, \eta^k \wedge \eta^\ell \\ \text{(Skew-symmetry)} \qquad 0 \quad &= \quad R^i_{jk\ell} + R^i_{j\ell k} \ = \ R^i_{jk\ell} + R^j_{ik\ell} \\ \text{(First Bianchi Identity)} \qquad 0 \quad &= \quad R^i_{jk\ell} + R^i_{k\ell j} + R^i_{\ell jk} \,. \end{aligned}$$

HW 10.8. Define $R_{ijk\ell} := R^i_{jk\ell}$. Prove that

$$\text{(Symmetry)} \qquad R_{ijk\ell} \ = \ R_{k\ell ij} \,.$$

(Hint: Define $S_{ijk\ell} = R_{ijk\ell} + R_{i\ell jk} + R_{ik\ell j}$ (this is the right-hand side of the Bianchi identity above), and consider $0 = S_{ijk\ell} + S_{jk\ell i} + S_{k\ell ij} + S_{\ell ijk}$.)

HW 10.9. **(a)** Given $u, \tau u \in \pi^{-1}(x)$ there exists $A \in \mathrm{O}(n) = \mathrm{O}(T_x M)$ such that $\tau u = u \cdot A = \mathcal{R}_A(u)$. In class we proved that

$$\mathcal{R}_A^* \eta_{\tau u} \ = \ A^{-1} \circ \eta_u \quad \text{and} \quad \mathcal{R}_A^* \vartheta_{\tau u} \ = \ \mathrm{Ad}_{A^{-1}}(\vartheta_u) \,.$$

Prove that $\mathcal{R}_A^* \Omega_{\tau u} = \mathrm{Ad}_{A^{-1}} \Omega_u$.

(b) Notice that $\mathcal{R}_A^* R^i_{jk\ell} = R^i_{jk\ell} \circ \mathcal{R}_A$, so that $(\mathcal{R}_A^* R^i_{jk\ell})_u = R^i_{jk\ell}(u \cdot A) = R^i_{jk\ell}(\tau u)$. Prove that $R^i_{jk\ell}(\tau u) = (A^{-1})^i_a R^a_{bcd}(u) A^b_j A^c_k A^d_\ell$.

(c) The fixed basis $\{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ of $\mathbb{R}^n$ defines a coframing $\{e^i = e^i(u)\}$ of $T_x M$ by $u = e^i\, \mathsf{v}_i$. Let $\{e_i = e_i(u)\}$ denote the dual framing. Prove that

$$\begin{aligned} e_k \wedge e_\ell \quad &\mapsto \quad R^i_{jk\ell}(u)\, e_i \otimes e^j \\ \tau e_k \wedge \tau e_\ell \quad &\mapsto \quad R^i_{jk\ell}(\tau u)\, \tau e^i \otimes \tau e_j \end{aligned}$$

define the *same* map $\bigwedge^2 T_x M \to \mathfrak{o}(T_x M)$. This map is the *curvature endomorphism at* x $R(x)$.

HW 10.10. Given a 2-plane $P \subset T_x M$, prove that the sectional curvature $K(P)$ does not depend on the choice of orthonormal basis of P.

HW 10.11. Define

$$\mathrm{Euc}(n) \ := \ \left\{ \left(\begin{array}{cc} 1 & 0 \\ x & e \end{array} \right) \ \middle| \ x \in \mathbb{R}^n \,,\ e \in \mathrm{O}(n) \right\} \,.$$

Observe that $\mathrm{Euc}(n)$ is a Lie group. Let

$$\mathfrak{euc}(n) \ = \ \left\{ \left(\begin{array}{cc} 0 & 0 \\ \xi & X \end{array} \right) \ \middle| \ \xi \in \mathbb{R}^n \,,\ X \in \mathfrak{so}(n) \right\}$$

denote the Lie algebra of $\mathrm{Euc}(n)$, and let $\varphi \in \Omega^1(\mathrm{Euc}(n), \mathfrak{euc}(n))$ denote the left-invariant Maurer-Cartan form on $\mathrm{Euc}(n)$.

Define a projection $\pi : \mathrm{Euc}(n) \to \mathbb{R}^n$ by $\pi(x, e) = x$. Observe that $(\mathrm{Euc}(n), \pi)$ is the $\mathrm{O}(V) \simeq \mathrm{O}(n)$–structure $\mathcal{G}$ of orthonormal frames on $M = \mathbb{R}^n$. Define $\eta = (\eta^i) \in \Omega^1(\mathrm{Euc}(n), \mathbb{R}^n)$ and $\vartheta = (\vartheta^i_j) \in \Omega^1(\mathrm{Euc}(n), \mathfrak{so}(n))$ by

$$\varphi \ =: \ \left(\begin{array}{cc} 0 & 0 \\ \eta & \vartheta \end{array} \right) \,.$$

Prove the following:

(a) $\mathrm{d}x = \eta^i\, e_i$ on $\mathrm{Euc}(n)$;

(b) $\mathrm{d}e_i = \vartheta_i^j\, e_j$ on $\mathrm{Euc}(n)$;

(c) η is the Solder form on $\mathcal{G}$;

(d) ϑ is the Levi-Civita connection form on $\mathcal{G}$.

HW 10.12. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^{n+1}$ and let $\mathrm{O}(n+1)$ denote the Lie group of $(n+1)\times(n+1)$–matrices preserving the inner product. Let $\mathfrak{so}(n+1)$ denote the Lie algebra of skew-symmetric matrices and let $\varphi \in \Omega^1(\mathrm{O}(n+1), \mathfrak{so}(n+1))$ denote the left-invariant Maurer-Cartan form.

Given an element $e = (e_0, \ldots, e_n) \in \mathrm{O}(n+1)$, define a projection $\pi : \mathrm{O}(n+1) \to S^n \subset \mathbb{R}^{n+1}$ by $\pi(e) = e_0$. Note that $(\mathrm{O}(n+1), \pi)$ is the $\mathrm{O}(n)$–structure $\mathcal{G}$ of orthonormal frames on S^n. Define $\eta = (\eta^i) \in \Omega^1(\mathrm{O}(n+1), \mathbb{R}^n)$ and $\vartheta = (\vartheta^i_j) \in \Omega^1(\mathrm{O}(n+1), \mathfrak{so}(n))$ by

$$\varphi \ =: \ \begin{pmatrix} 0 & -\eta \\ \eta & \vartheta \end{pmatrix} .$$

Prove the following:

(a) $\mathrm{d}e_i = \varphi_i^j\, e_j$ for $0 \le i, j \le n$;

(b) η is the Solder form on $\mathcal{G}$;

(c) ϑ is the Levi-Civita connection form on $\mathcal{G}$.

HW 10.13. Fix a basis $\{\mathsf{v}_0, \mathsf{v}_1, \ldots, \mathsf{v}_n\}$ of $\mathbb{R}^{n+1}$. Let $x = (x^0, x^1, \ldots, x^n)$ be the induced coordinates on $\mathbb{R}^{n+1}$. Define a nondegenerate, symmetric, bilinear form Q on V by

$$Q(x,y) \ = \ -x^0 y^0 \ + \ \sum_{a=1}^n x^a y^a \, .$$

Let $\mathrm{O}(1,n) = \{A \in \mathrm{GL}(n+1) \mid Q(Ax, Ay) = Q(x,y)\}$. Note that $\mathrm{O}(1,n)$ is a Lie group. Let $\mathfrak{so}(1,n)$ denote the Lie algebra and let $\varphi \in \Omega^1(\mathrm{O}(1,n), \mathfrak{so}(1,n))$ denote the left-invariant Maurer-Cartan form. Prove the following:

(a) An element $e = (e_0, e_1, \ldots, e_n)$ of $\mathrm{GL}(n{+}1)$ lies in $\mathrm{O}(1,n)$ if and only if $Q(e_0, e_0) = -1$, $Q(e_a, e_b) = \delta_{ab}$ for all $1 \le a, b \le n$, and $Q(e_i, e_j) = 0$ for all $0 \le i \ne j \le n$.

(b) The Lie algebra is

$$\mathfrak{so}(1,n) \ = \ \left\{ X = \begin{pmatrix} 0 & X^0_b \\ X^a_0 & X^a_b \end{pmatrix} \in \mathfrak{gl}(n+1) \ \middle| \ X^a_0 = X^0_a \, , \ X^a_b = -X^b_a \right\} .$$

(c) $\mathrm{d}e_i = \varphi_i^j\, e_j$ on $\mathrm{O}(1,n)$.

HW 10.14 (Continuation of HW 10.13)**.** Let $M^n = \{x \in \mathbb{R}^{n+1} \mid Q(x,x) = -1\}$. Then M is a smooth hypersurface in $\mathbb{R}^{n+1}$. Define a metric g on M by restriction of Q: given $u, v \in T_xM \subset \mathbb{R}^{n+1}$, set $g(u,v) = Q(u,v)$.

(a) Show that given any $x \in M$ there exists $e \in \mathrm{O}(1,n)$ with $e_0 = x$. (This implies that $\mathrm{O}(1,n)$ acts transitively on M.)

(b) Prove that g is a Riemannian metric on M: symmetric and positive definite. (Hint: show that T_xM may be identified with $\{v \in \mathbb{R}^{n+1} \mid Q(x,v) = 0\}$.)

HW 10.15 (Continuation of HW 10.14)**.** Given $e = (e_0, e_1, \ldots, e_n) \in \mathrm{O}(1,n)$ define $\pi : \mathrm{O}(1,n) \to M$ by $\pi(e) = e_0$. Note that $(\mathrm{O}(1,n), \pi)$ is the $\mathrm{O}(n)$–structure $\mathcal{G}$ of orthonormal frames on M. Define $\eta = (\eta^i) \in \Omega^1(\mathrm{O}(1,n), \mathbb{R}^n)$ and $\vartheta = (\vartheta^i_j) \in \Omega^1(\mathrm{O}(1,n), \mathfrak{so}(n))$ by

$$\varphi =: \begin{pmatrix} 0 & \eta \\ \eta & \vartheta \end{pmatrix} .$$

Prove the following:

- **(a)** η is the Solder form on $\mathcal{G}$;
- **(b)** ϑ is the Levi-Civita connection form on $\mathcal{G}$;
- **(c)** (M, g) has constant sectional curvature $K = -1$.

11. Principle bundles

Definition. Let M be a manifold and G a Lie group. A *principle G–bundle over M* consists of a manifold $P = P(M, G)$ and a right-action

$$P \times G \to P\,, \qquad (u, a) \mapsto ua =: \mathcal{R}_a u\,,$$

of G on P satisfying the following conditions:

- *(a)* G acts freely: $u = ua \iff a = \mathrm{id} \in G$;
- *(b)* $M = P/G$ and the quotient map $\pi : P \to M$ is differentiable;
- *(c)* P is locally trivial. That is, for all $x \in M$ there exists a neighborhood $U \subset M$ and a smooth map $\varphi : \pi^{-1}(U) \to G$ with the property that $u \mapsto (\pi(u), \varphi(u))$ defines a diffeomorphism $\pi^{-1}(U) \to U \times G$, and $\varphi(ua) = \varphi(u)a$.

Let $P_x := \pi^{-1}(x)$ denote the *fibre over $x \in M$.*

Remark. A G–structure on M is an example of a principle bundle.

Definition. A *homomorphism* $f : P(M, G) \to Q(N, H)$ *of principle bundles* is a pair of mappings (both denoted f!):

$$\begin{array}{ll} f : P \to Q & \text{a smooth map of manifolds} \\ f : G \to H & \text{a homomorphism of groups} \end{array}$$

such that $f(ua) = f(u)f(a)$ for all $u \in P$ and $a \in G$.

HW 11.1. Let $f : P(M, G) \to Q(N, H)$ be a homomorphism of vector bundles.

- *(a)* Prove that f preserves fibres. That is, given $x \in M$, there exists $y \in N$ such that $f(P_x) \subset Q_y$.
- *(b)* Prove that f induces a well-defined map $f : M \to N$, and show that this map is smooth.

Definition. Let $V_u := \ker\{\pi_* : T_uP \to T_xM\,,\ x = \pi(u)\}$ denote the *vertical subspace at* $u \in P$. Given $X \in \mathfrak{g}$, let $\widetilde{X} \in \mathfrak{X}(P)$ denote the induced vertical vector field

$$\widetilde{X}_u := \frac{\mathrm{d}}{\mathrm{d}t} u \exp(tX)\Big|_{t=0} .$$

Definition. A *connection* on P is a choice of horizontal subspace $H_u \in T_uP$ at every $u \in P$ such that

- *(a)* $T_uP = V_u \oplus H_u$, and
- *(b)* $(\mathcal{R}_a)_* H_u = H_{ua}$.

Let $h : T_uP \to H_u$ denote the projection onto the horizontal subspace.

Every connection defines a *connection* 1*-form* $\vartheta \in \Omega^1(P, \mathfrak{g})$ by

$$\vartheta(\widetilde{X}) \ := \ X \quad \forall X \in \mathfrak{g} \qquad \text{and} \qquad \vartheta(\zeta) \ := \ 0 \quad \forall\, \zeta \in H\,.$$

HW 11.2. Prove that $(\mathcal{R}_a)_* \circ h = h \circ (R_a)_*$.

HW 11.3. Prove that $(\mathcal{R}_a)^* \vartheta = \mathrm{Ad}_{a^{-1}} \vartheta$.

Definition. Let P be a principle bundle with a connection. The *curvature of the connection* is the 2-form $\Omega \in \Omega^2(P, \mathfrak{g})$ defined by

$$\Omega \ := \ \mathrm{d}\vartheta\, h\,.$$

The curvature satisfies the *structure equation*

$$\mathrm{d}\vartheta \ + \ \tfrac{1}{2}[\vartheta, \vartheta] \ = \ \Omega\,.$$

Above $[\vartheta, \vartheta] \in \Omega^2(P, \mathfrak{g})$ is the 2-form defined by $[\vartheta, \vartheta](V, W) := 2[\vartheta(V), \vartheta(W)]$.

HW 11.4. Prove that $\Omega(V, W) = -\vartheta([V, W])$ for all horizontal $V, W \in \mathfrak{X}(P)$.

Definition. Fix a connection on P. Let $u \in P$ and $x = \pi(u)$. Every curve $\gamma : [0,1] \to M$ with $\gamma(0) = x$ has a unique *horizontal lift* $\bar\gamma_u : [0,1] \to P$ *to* u. The lift is determined by

$$\bar\gamma_u(0) \ = \ u\,, \quad \pi \circ \bar\gamma_u \ = \ \gamma\,, \quad \text{and} \ \ \bar\gamma_u' \in H_{\bar\gamma_u}\,.$$

Given $u, v \in P$, we write $u \sim v$ when $v = \bar\gamma_u(1)$ for some curve γ.

Notation. Given $x \in M$, let $C(x) = \{\gamma : [0,1] \to M \mid \gamma(0) = x = \gamma(1)\}$ denote the *closed curves at* x. Let $C^0(x) \subset C(x)$ denote the closed curves at x that are homotopic to the constant curve $\gamma_0(t) = x$.

Let $\gamma^{-1} : [0,1] \to M$ denote the curve $\gamma^{-1}(t) = \gamma(1-t)$. Given a second closed curvature $\beta : [0,1] \to M$ at x define a $\beta \cdot \gamma : [0,1] \to M$ by

$$\beta \cdot \gamma(t) \ := \ \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \beta(2t-1) & t \in [\frac{1}{2}, 1]\,. \end{cases}$$

Definition. Given $\gamma \in C(x)$ the *parallel displacement by* γ is the map $\tilde\gamma : P_x \to P_x$ defined by $u \mapsto \bar\gamma_u(1)$.

HW 11.5. Using the notation above, prove the following.

- *(a)* Given $a \in G$, $\mathcal{R}_a \circ \tilde\gamma = \tilde\gamma \circ \mathcal{R}_a$.
- *(b)* $\widetilde{\gamma^{-1}} = \tilde\gamma^{-1}$.
- *(c)* $\widetilde{\beta \cdot \gamma} = \tilde\beta \circ \tilde\gamma$.

Definition. If $P = P(M,G)$ is a principle bundle with connection Γ, the *holonomy group of* Γ *with reference point* $x \in M$ is

$$\Phi(x) \ := \ \{\tilde{\gamma} : P_x \to P_x \mid \gamma \in C(x)\}\,.$$

The *holonomy group of* Γ *with reference point* $u \in P$ is

$$\Phi(u) \ := \ \{a \in G \mid u \sim ua\} \ \subset \ G\,.$$

Definition. A *reduction* of $P = P(M,G)$ is a principle bundle $P' = P(M,H)$ and a vector bundle homomorphism $f : P' \to P$ such that

(a) $f : P' \to P$ is an embedding of manifolds;

(b) $f : H \to G$ is a monomorphism;

(c) $f : M \to M$ is the identity.

If Γ and Γ' are connections on P and P', with connection forms $\vartheta \in \Omega^1(P,\mathfrak{g})$ and $\vartheta' \in \Omega^1(P',\mathfrak{h})$, then Γ *is reducible to* Γ' if

$$f^*\vartheta \ = \ f \cdot \vartheta'\,. \tag{11.1}$$

(The group homomorphism $f : H \to G$ defines $\mathrm{d}f_e : \mathfrak{h} \to \mathfrak{g}$. The $f \cdot \vartheta$ above is $\mathrm{d}f_e(\vartheta)$.)

HW 11.6. Assume (11.1) holds. Set $u = f(u')$. Prove the following:

(a) $f^*\Omega_u = f \cdot \Omega'_{u'}$.

(b) $f_* H'_{u'} = H_u$.

(c) Let $\gamma : [0,1] \to M$ be a curve such that $\gamma(0) = x = \pi'(u') = \pi(u)$. Let $\bar{\gamma}_u$ be the horizontal lift of γ in M to $u \in P$, and let $\bar{\gamma}_{u'}$ be the horizontal lift of γ to $u' \in P'$. Prove that $f \circ \bar{\gamma}_{u'} = \bar{\gamma}_u$.

(d) $f(\Phi(u')) = \Phi(u)$.

12. Integration

Definition. A n-dimensional manifold M is *orientable* if and only if there exists a no-where zero n-form $\mu \in \Omega^n(M)$: $\mu_p \neq 0 \in \bigwedge^n T_p^* M$ for all $p \in M$.

HW 12.1 (Orientability and coordinate charts)**.** Prove that M is orientable if and only if M can be covered by coordinate charts $\{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ with the property that $\det(\frac{\partial x_\alpha}{\partial x_\beta}) > 0$ on $U_\alpha \cap U_\beta$.

Hint. Use partitions of unity to construct an orientation $\mu \in \Omega^n(M)$ for the direction ($\Longleftarrow$).

Definition. Set

$$H^n \ = \ \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^1 \leq 0\}\,.$$

Given $H^n \subset \mathbb{R}^n$ the subspace topology: the open sets $W \subset H^n$ are of the form $W = U \cap H^n$ with $U \subset \mathbb{R}^n$ open. A function $f : H^n \to \mathbb{R}$ is *differentiable at* $p \in H^n$ if there exists a differentiable function $\overline{f} : U \to \mathbb{R}$ defined open neighborhood $U \subset \mathbb{R}^n$ of p such that $f = \overline{f}$ on $U \cap H^n$.

Definition. A topological space M is an *n-dimensional manifold with boundary* if M admits an open covering $\{U_\alpha\}$ and homeomorphisms $x_a : U_a \to W_\alpha$ onto open sets $W_\alpha \subset H^n$ such that the $x_\beta \circ x_\alpha^{-1}$ are differentiable. The *boundary ∂M of M* is

$$\partial M \;=\; \{p \in M \mid \exists\, (U,x) \text{ s.t. } x^1(p) = 0\}\,.$$

That is, some coordinate chart (U,x) maps p into the boundary of H^n.

HW 12.2. Assume that $x^1(p) = 0$ for some coordinate chart (U,x) on M. Let (V,y) be any other coordinate chart containing p. Prove that $y^1(p) = 0$. (That is, if $p \in M$ is mapped into ∂H^n by *some* coordinate chart, then p is mapped into ∂H^n by *every* coordinate chart.)

HW 12.3. Let M^n be a compact orientable manifold without boundary ($\partial M = \emptyset$). Let $\omega \in \Omega^{n-1}(M)$. Prove that there exists a point $p \in M$ such that $\mathrm{d}\omega_p = 0$.

HW 12.4 (Area in $\mathbb{R}^2$)**.** Let $\mathbb{R}^2$ have Cartesian coordinates (x,y). Set $\omega = x\mathrm{d}y - y\mathrm{d}x \in \Omega^1(\mathbb{R}^2)$. Assume that $i : M \hookrightarrow \mathbb{R}^2$ is a bounded region with regular boundary ∂M. (That is, M is a compact 2-dimensional submanifold of $\mathbb{R}^2$ with boundary ∂M. For example, $M = \{x \in \mathbb{R}^2 \mid |x| \le 1\}$ and $\partial M = S^1$.) Prove that

$$\mathrm{Area}(M) \;=\; \tfrac{1}{2}\int_{\partial M} i^*\omega\,.$$

HW 12.5 (Green's identities)**.** Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be differentiable functions. Let $M^3 \subset \mathbb{R}^3$ be a compact differentiable manifold with boundary ∂M^2. Let $\nu \in \Omega^3(M)$ denote the volume element of M. Given $p \in \partial M$, let N_p denote the outward pointing unit normal of ∂M, so that $\sigma = N \lrcorner\, \nu$ is the volume element on ∂M.

(a) First identity: Prove that

$$\int_M \langle \mathrm{grad}(f), \mathrm{grad}(g)\rangle\, \nu \;+\; \int_M f\,\Delta g\,\nu \;=\; \int_{\partial M} f \langle \mathrm{grad}(g), N\rangle\, \sigma\,,$$

where $\Delta f = \sum_{i=1}^3 \frac{\partial^2 f}{\partial (x^i)^2}$ is the Laplacian. [Hint: set $v = f\,\mathrm{grad}(g)$ in the Divergence Theorem.]

(b) Second identity: Prove that

$$\int_M (f\,\Delta g \,-\, g\,\Delta f)\,\nu \;=\; \int_{\partial M} \big(f\,\langle \mathrm{grad}(g), N\rangle \,-\, g\,\langle \mathrm{grad}(f), N\rangle\big)\,\sigma\,.$$

13. deRHAM COHOMOLOGY

13.1. Poincaré's Lemma.

Definition. A manifold M is *contractible* (to a point $p_o \in M$) if there exists a smooth map $H : M \times \mathbb{R} \to M$ such that

$$\begin{aligned} H(p,1) &= p \quad \forall\, p \in M\,,\\ H(p,0) &= p_o \quad \forall\, p \in M\,. \end{aligned}$$

Definition. Let $\pi : M \times \mathbb{R} \to M$ denote the projection $\pi(p,t) = p$. A vector $v \in T_{(p,t)}(M \times \mathbb{R})$ is *vertical* if $\pi_*(v) = 0 \in T_pM$. A k-form $\zeta \in \Omega^k(M \times \mathbb{R})$ is *semi-basic* if

$$\zeta(v_1, \ldots, v_k) \;=\; 0$$

whenever any of the v_i are vertical.

HW 13.1. Prove that $\mathbb{R}^n$ and $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$ are contractible.

HW 13.2. Prove that the vertical vectors are spanned by $\frac{\partial}{\partial t}$.

HW 13.3. In this exercise you will confirm the assertions made in Lemma 1 of the the proof of Poincaré's Lemma. Given $\zeta \in \Omega^k(M \times \mathbb{R})$, define $\eta_1 = \frac{\partial}{\partial t} \lrcorner \zeta$ and $\eta_0 = \zeta - \mathrm{d}t \wedge \eta_1$. Prove that

- **(a)** The forms η_0 and η_1 are semi-basic.
- **(b)** $\zeta = \eta_0 + \mathrm{d}t \wedge \eta_1$.
- **(c)** The forms η_0 and η_1 are the *unique* semi-basic forms such that (b) holds.

13.2. **deRham cohomology.**

HW 13.4. Let $F : M \to N$ be a smooth map on manifolds. The pull-back $F^* : \Omega^k(N) \to \Omega^k(M)$ induces a map $F^* : H^k(N) \to H^k(N)$ by $F^*([\omega]) := [F^*\omega]$. Prove that the induced map is a well-defined linear map of vector spaces.

HW 13.5. Suppose that $F : M \to N$ is a diffeomorphism. Prove that $F^* : H^k(N) \to H^k(M)$ is a vector space isomorphism.

HW 13.6. Prove that the 2-sphere S^2 is not contractible.

HW 13.7. Compute the cohomology of $M = \mathbb{R}^2 \backslash \{(\pm 1, 0)\}$.

HW 13.8 (MT 5.3)**.** Can $\mathbb{R}^2$ be written as the union $\mathbb{R}^2 = U_1 \cup U_2$ of two connected open sets such that $U_1 \cap U_2$ is disconnected? (I.e. the intersection consists of more than one connected component.)

13.3. **Harmonic forms.**

Assume. M is an n-dimensional oriented Riemannian manifold. Let $\nu \in \Omega^n(M)$ denote the volume form: given any oriented, orthonormal basis $v_1, \ldots, v_n\}$ of T_xM, $\nu_x(v_1, \ldots, v_n) = 1$.

Notation. Let $\{\omega^1, \ldots, \omega^n\}$ be a oriented, orthonormal coframing of M, locally defined on an open set $U \subset M$. Recall that $\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \mid i_1 < \cdots < i_p\}$ forms a basis, over $\mathcal{C}^\infty(U)$, of $\Omega^p(U)$.

Remark. On U, $\nu = \omega^1 \wedge \cdots \wedge \omega^n$.

Notation. Given $I = \{i_1, \ldots i_p\} \subset \{1, \ldots, n\}$, let

$$\omega^I \ := \ \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \,.$$

Definition. Define the *Hodge star operator* $* : \Omega^p(U) \to \Omega^{n-p}(U)$ as follows. Let $I^c = \{i_{p+1}, \ldots, i_n\}$ be the complement of I in $\{1., \ldots, n\}$. Define

$$*\omega^I \ := \ \begin{cases} \omega^{I^c} & \text{if } \omega^I \wedge \omega^{I^c} = \nu, \\ -\omega^{I^c} & \text{if } \omega^I \wedge \omega^{I^c} = -\nu. \end{cases}$$

This defines the Hodge star on a basis of $\Omega^p(U)$. Extend the definition to all of $\Omega^p(U)$ by requiring that $*$ be $\mathcal{C}^\infty(U)$–linear.

Remark. Given $I = \{i_1 < \cdots < i_p\}$ and $I^c = \{i_{p+1} < \cdots < i_n\}$, let $\mathrm{sgn}(I) = \pm 1$ be the sign of the permutation mapping $k \mapsto i_k$. Then $*\omega^I = \mathrm{sgn}(I)\omega^{I^c}$.

Definition. Define an inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^p T_x^* M$ by declaring the basis vectors $\{\omega_x^I : |I| = p\}$ to be an *orthonormal* basis of $\bigwedge^p T_x^* M$.

HW 13.9 (Use the definitions above.)**.**

(a) Let $\alpha_x = A_I \omega_x^I$ and $\beta_x = B_J \omega_x^J$ in $\bigwedge^p T_x^* M$. Prove that $\langle \alpha_x, \beta_x \rangle = \sum A_I B_I$.

(b) Given $\alpha_x \in \bigwedge^p T_x^* M$ and $\gamma_x \in \bigwedge^{n-p} T_x^* M$, prove that $\gamma_x = *\alpha_x$ if and only if $\alpha_x \wedge \beta_x = \langle \gamma_x, \beta_x \rangle \, \nu$ for all $\beta_x \in \bigwedge^{n-p} T_x^* M$.

(c) Prove that the definition of the Hodge star operator $* : \bigwedge^p T_x^* M \otimes \bigwedge^{n-p} T_x^* M$ does not depend on our choice of oriented, orthonormal coframing $\{\omega_x^1, \ldots, \omega_x^n\}$. (In particular, the definition of the Hodge star operator above extends to a well-defined map $* : \Omega^p(M) \to \Omega^{n-p}(M)$.)

Hint. It is a consequence of part (b) above that $*\alpha_x$ may be defined (or characterized) by $\alpha_x \wedge \beta_x =: \langle *\alpha_x, \beta_x \rangle$ for all $\beta_x \in \bigwedge^{n-p} T_x^* M$. You may find this helpful in proving part (c).

HW 13.10. Prove that $** : \Omega^p(M) \to \Omega^p(M)$ is $(-1)^{p(n-p)}\,\mathrm{Id}$.

HW 13.11. Suppose that M is Euclidean space $\mathbb{R}^n$. Prove that $\Delta = -\sum_{i=1}^n \partial^2 / \partial {x_i}^2$ on functions.

HW 13.12. Prove that $*\Delta = \Delta *$.

Assume. M is compact.

Definition 13.1. Given $\alpha, \beta \in \Omega^p(M)$ define

$$(\alpha, \beta) \ := \ \int_M \alpha \wedge *\beta \, .$$

HW 13.13. **(a)** Prove that $(\cdot, \cdot)$ is bilinear. That is, given $\alpha, \beta, \gamma \in \Omega^p(M)$ and $r \in \mathbb{R}$,

(i) $(\alpha, \beta + \gamma) = (\alpha, \beta) + (\alpha, \gamma)$ and $(\alpha + \gamma, \beta) = (\alpha, \beta) + (\gamma, \beta)$,

(ii) $(r\alpha, \beta) = r\,(\alpha, \beta) = (\alpha, r\,\beta)$.

(c) Prove that $(\cdot, \cdot)$ is symmetric and positive definite. That is, $(\alpha, \beta) = (\beta, \alpha)$ and $(\alpha, \alpha) > 0$ for all nonzero α.

Hint. HW 13.9.

Remark. The exercise above implies that $(\cdot, \cdot)$ is an inner-product on the (infinite dimensional) vector space of p-forms.

HW 13.14. Prove that $\delta^2 = 0$.

HW 13.15. Suppose that $\alpha, \beta \in \Omega^n(M)$ and $\int_M \alpha = \int_M \beta$. Assume that M is connected and prove that $\alpha - \beta$ is exact.

HW 13.16. Prove that G is a bounded, self-adjoint linear operator.

14. GAUSS-BONNET THEOREM

HW 14.1. Prove that every smooth vector field on the 2-sphere has a zero.

15. Euler class

HW 15.1. Compute the Euler class of the torus T^{2m}. (That is, the Euler class of the tangent bundle.)

HW 15.2. Consider the tautological line bundle $\mathcal{T}$ over complex projective n-space $\mathbb{CP}^n$. Recall that $\mathcal{T}_x$ naturally inherits a Hermitian metric from $\mathbb{C}^{n+1}$. We used the curvature of the corresponding Chern connection to compute the Chern classes $c_k(\mathcal{T})$.

If we ignore the complex structure on the fibres, we may regard $\mathcal{T}$ as a real, rank 2 vector bundle E. Prove that E is oriented and compute the Euler class of E in terms of $c_1(\mathcal{T})$.

HW 15.3 (Thom Isomorphism)**.** Let E be a real, oriented, rank k vector bundle over an oriented, connected, compact manifold M with Thom class $\mathbf{U}$. Prove that the map $H^\ell(E) \to H_c^{\ell+k}(E)$ sending $\alpha \mapsto \alpha \wedge \mathbf{U}$ is an isomorphism for all ℓ.

HW 15.4. Suppose that E and F are oriented, real vector bundles over a closed oriented manifold M. Prove that $\mathbf{U}_{E\oplus F} = \mathbf{U}_E \mathbf{U}_F$.