Harmonic form HW from lecture

HW Asm V is an oriented innerproduct space with oriented orthonormal basis $\{e_1, \dots, e_n\}$. Let $\eta \in \Lambda^k V^*$. Prove that
$$(*\eta)(e_{k+1}, \dots, e_n) = \eta(e_1, \dots, e_k).$$

Prf: Let $\{\omega^1, \dots, \omega^n\}$ be a basis of V^* dual to the $\{e_i\}_{i=1}^n$. By the linearity of $*: \Lambda^k V^* \longrightarrow \Lambda^{n-k} V^*$, it suffices to establish the claim for $\eta = \omega^{i_1} \wedge \dots \wedge \omega^{i_k} = \omega^I$ with $\{1 \leq i_1 < \dots < i_k \leq n\} = I$.

Let $J = \{1 \leq j_1 < \dots < j_{n-k} \leq n\}$ denote the compliment of I in $\{1, \dots, n\}$.

Then $*\omega^I = \pm \omega^J$.

Observe that both $(*\omega^I)(e_{k+1}, \dots, e_n)$ and $\omega^I(e_1, \dots, e_k)$ are zero unless $I = \{1, \dots, k\}$. Assume this is the case. Then $*\omega^I = \omega^J$ and
$$(*\eta)(e_{k+1}, \dots, e_n) = 1 = \eta(e_1, \dots, e_k). \quad \blacksquare$$

Harmonic form HW from lecture: cpt Lie gps.

Asm Assume throughout that G is a compact Lie group with bi-invariant metric g. Let $\nu = da$ denote the associated Haar measure.

$\boxed{\text{HW}}$ Define $L : \Omega^k \to \Omega^k$ by $L(\eta) = \int_G L_a^* \eta \, da$

Prove (a) $L\eta$ is left invariant

(b) If η is left-invariant, then $L\eta = \eta$.

Prf: Fix $b \in G$. Compute

left-invar of Haar meas.

$$L_b^*(L\eta) = L_b^* \int_G L_a^* \eta \, da$$

$$= \int_G L_{ba}^* \eta \, da = \int_G L_a^* \eta \, da = L\eta.$$

This establishes (a).

Haar measure has unit volume.

Spse η is left-invar. Then

$$L\eta = \int_G L_a^* \eta \, da = \int_G \eta \, da = \eta \cdot \int_G da = \eta \cdot 1$$

$$= \eta.$$

∎

HW Define $I : \Omega^k \to \Omega^k$ by $I\eta = \int_G R_a^*(L\eta)\, da$

Prove (a) $I\eta$ is bi-invar.

(b) if η is bi-invar. then $I\eta = \eta$.

Proof: Fix $b \in G$. Compute

right-invar of Haar measure

- $$R_b^*(I\eta) = \int_G R_{ab}^*(L\eta)\, da = \int_G R_a^*(L\eta)\, da = I\eta$$

- $$L_b^*(I\eta) = \int_G R_a^* L_b^*(L\eta)\, da$$

left-invar of $L\eta$

$$= \int_G R_a^*(L\eta)\, da = I\eta$$

Conclude $I\eta$ is bi-invar.

Asm η is bi-invariant. Then

prev. exercise & left-invar of η

$$I\eta = \int_G R_a^*(L\eta)\, da = \int_G R_a^* \eta\, da$$

$$= \int_G \eta\, da = \eta \cdot \int_G da = \eta$$

right-invar of η.

∎

(Additional asm: G is connected)

HW Assume that $\eta \in \Omega^k$ is bi-invariant. Prove η is co-closed ($d^*\eta = 0$)

Prf: In class we proved that every bi-invariant form is closed! Since $d^*\eta = \pm * d * \eta$, it suffices to show that $*\eta$ is bi-invariant.

Fix an oriented orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g} = T_{\mathbb{1}}G$. Because the metric on G is left-invariant, $\{(L_a)_* e_j =: E_j\}_{j=1}^n$ is an orthonormal basis of T_aG. Because G is connected, $\{E_j\}_{j=1}^n$ is an oriented basis. Let $\{E_j\}_{j=1}^n$ denote the associated positive, orthonormal, left-invariant framing of G.

Prev. HW exercise.

Then $(*\eta)(E_{k+1}, \ldots, E_n) = \eta(E_1, \ldots, E_k)$

From the left-invar of η and $\{E_j\}_{j=1}^n$, it follows that $*\eta$ is left-invar.

Next $(R_a^*(*\eta))(E_{k+1}, \ldots, E_n)$

$= (*\eta)(R_{a*}E_{k+1}, \ldots, R_{a*}E_n)$ (1)

$= \eta(R_{a*}E_1, \ldots, R_{a*}E_k)$ (2)

The equality of (1) & (2) is a conseq of the fact that R_{a*} preserves the metric and the orientation (again G connected).

Finally, the right invar of η $\Rightarrow$

$$(2) = \eta(E_1, \dots, E_k)$$
$$= (*\eta)(E_{k+1}, \dots, E_n).$$

Thus $*\eta$ is right-invar.

Conclude $*\eta$ is bi-invar. $\blacksquare$