Jost. Ch 2.

#2 Consider the action of $SO(3)$ on $S^2 \subset \mathbb{R}^3$.
Let $\omega \in \Omega^1(S^2)$ and spse $\varphi^* \omega = \omega$
$\forall \varphi \in SO(3)$.

Claim: $\omega = 0$.

Proof: Fix $p \in S^2$. It suffices to show that $\omega_p = 0 \in T_p^* S^2$.

Let $Stab_p \subset SO(3)$ denote the stabilizer of p. Note that $Stab_p \simeq SO(2)$ acts transitively on the unit circle $S^1 \subset T_p S^2$.

If $\omega_p(\xi) \neq 0$ for some $\xi \in T_p S^2$, then the facts

1) $\varphi^* \omega_p = \omega_p \quad \forall \varphi \in Stab_p$
2) $Stab_p$ acts transitively on $S^1 \subset T_p S^2$

imply $\omega_p(\xi) \neq 0 \quad \forall\, 0 \neq \xi \in T_p S^2$.

This contradicts the fact that every $\omega_p \in T_p^* S^2$ vanishes on a codim $= 1$ subspace of $T_p S^2$ ∎

#3 Let $n = \dim M$. <u>On $\Omega^k(M)$</u> :

(1) $*^2 = (-1)^{k(n-k)}$

(2) $d^* = (-1)^{n(k+1)+1} * d *$.

(3) $\Delta = d^* d + d d^*$.

<u>So</u> :

- $$\Delta * = d^* \overbrace{d *}^{n-k+1} + d d^* *$$
$$= (-1)^{n(n-k+2)+1} * d * d * + (-1)^{n(n-k+1)+1} d * d *^2$$
$$= (-1)^{n(n-k)+1} * d * d * + (-1)^{k^2+1} d * d$$

- $$* \Delta = * d^* d + * d d^*$$
$$= (-1)^{nk+1} *^2 \overbrace{d * d}^{n-k} + (-1)^{k(n-k)} * d * d *$$
$$= (-1)^{k^2+1} d * d + (-1)^{k(n-k)} * d * d * .$$

It follows from two comps above that $* \Delta = \Delta *$ on $\Omega^k(M)$.

On $\Omega^1(\mathbb{R}^2)$: $\Delta = d^*d + dd^*$
$$= -*d*d - d*d*$$

So $\Delta(P_1 dx^1 + P_2 dx^2) =$

$$-*d*(P_{21} - P_{12})\, dx^1 \wedge dx^2 - d*d(P_i H^i{}_j\, dx^j)$$

$$= -*d[(P_{21} - P_{12}) \det(A)]$$

$$- d*(H^i{}_j\, dP_i \wedge dx^j + P_i\, dH^i{}_j \wedge dx^j)$$

$$= - \frac{\partial}{\partial x^i}[(P_{21} - P_{12}) \det(A)]\, H^i{}_j\, dx^j$$

$$- d\Big[\det(A)\big(H_2{}^i P_{i1} - H_1{}^i P_{i2}$$

$$+ P_i H^i{}_{21} - P_i H^i{}_{12}\big)\Big]$$

#4 Let $\{e_1, e_2\}$ be a local, oriented orthonormal frame.

$$e_i = A_i^{\ j} \frac{\partial}{\partial x^j}$$

$$1 = g(e_i, e_i) = A_i^{\ j} g_{jk} A^k_{\ i} \qquad i = 1, 2$$

$$0 = g(e_1, e_2) = A_1^{\ j} g_{jk} A^k_{\ 2}$$

Let $B = A^{-1}$ Then $\{\omega^i = B^i_{\ j}\, dx^j\}_{i=1,2}$

is dual to $\{e_i\}_{i=1,2}$.

Thus $* \omega^1 = \omega^2$ and $* \omega^2 = -\omega^1$

So $$* dx^1 = * A^1_{\ j} \omega^j = A^1_{\ 1} \omega^2 - A^1_{\ 2} \omega^1$$

$$= \left(A^1_{\ 1} B^2_{\ j} - A^1_{\ 2} B^1_{\ j}\right) dx^j =: H^1_{\ j} dx^j$$

and $$* dx^2 = * A^2_{\ j} \omega^j = A^2_{\ 1} \omega^2 - A^2_{\ 2} \omega^1$$

$$= \left(A^2_{\ 1} B^2_{\ j} - A^2_{\ 2} B^1_{\ j}\right) dx^j =: H^2_{\ j} dx^j$$

Also $*(dx^1 \wedge dx^2) = * \det(A)\, \omega^1 \wedge \omega^2 = \det(A)$

#7a Spse $\Delta u = \lambda u$. W.T.S. $\lambda \geq 0$

$$\lambda \langle u,u \rangle = \langle \Delta u, u \rangle$$

$$= \langle d^* du, u \rangle + \langle dd^* u, u \rangle$$

$$= \langle du, du \rangle + \langle d^* u, d^* u \rangle$$

$$= \| du \|^2 + \| d^* u \|^2 \geq 0.$$

#7b Assume the eigenspace $E(\lambda)$ for eigenvalue λ is not finite dim'l. Pick a sequence $\{u_i\}_{i=1}^{\infty}$ of pairwise orthonormal eigenvectors.

Then $\|u_i\| = 1$ and $\|\Delta u_i\| = \lambda$

Thm B (lecture) $\Rightarrow \exists$ a Cauchy subseq. Contradiction.

$E(\lambda)$ is finite dim'l.

#7c. Let $\{\lambda_j\}_{j=1}^{\infty}$ be a distinct seq of eigenvalues and suppose $\lambda_j \to \lambda_0 < \infty$

Let u_j be a unit eigenvector for λ_j.

That is $\Delta u_j = \lambda_j u_j$.

This proves #7d.

Note that $\lambda_j \langle u_j, u_k \rangle = \langle \Delta u_j, u_k \rangle$
$= \langle u_j, \Delta u_k \rangle = \lambda_k \langle u_j, u_k \rangle$ implies the $\{u_j\}$ are pairwise orthogonal.

Since $\|u_j\| = 1$ and $\|\Delta u_j\| = \lambda_j \to \lambda_0 < \infty$

Thm B $\Rightarrow$ $\{u_j\}$ has a Cauchy sub seq. Contradiction. $\therefore$ $\nexists$ no finite accumulation point λ_0 for the eigenvalues.