$\boxed{\text{MT 5.3}}$ No. Consider the long exact sequence

$$
\begin{array}{ccccccccc}
0 \to & H^0(\mathbb{R}^2) & \to & H^0(U) \oplus H^0(V) & \to & H^0(U \cap V) & \to & H^1(\mathbb{R}^2) & \to \cdots \\
& \| & & \| & & & & \| & \\
& \mathbb{R} & & \mathbb{R}^2 & & & & 0 &
\end{array}
$$

We obtain a short exact sequence

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to H^0(U \cap V) \to 0$$

$$\Rightarrow \quad H^0(U \cap V) = \mathbb{R}^2/\mathbb{R} = \mathbb{R}$$

$\Rightarrow$ $U \cap V$ is connected. $\blacksquare$

#5.4

Let $X \subset \mathbb{R}^n$ be a closed set. Then X separates p from q if and only if $\exists$ a locally constant $f: \mathbb{R}^n \setminus X \to \mathbb{R}$ s.t. $f(p) \neq f(q)$.

Set $U = \mathbb{R}^n \setminus A$ and $V = \mathbb{R}^n \setminus B$. Then $U \cap V = \mathbb{R}^n \setminus \{A \cup B\}$. Mayer-Vietoris $\rightsquigarrow$

$$0 \to H^0(U \cup V) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V)$$

$$\to H^1(U \cup V) \to \cdots$$

Since $U \cup V = \mathbb{R}^n$ the long exact M-V seq $\Rightarrow$

$$0 \to \mathbb{R} \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to 0$$

is a short exact seq.

Thus $H^0(U \cap V)$ = locally cnst fns on $\mathbb{R}^n \setminus \{A \cup\}$ is naturally identified with

$$\underbrace{H^0(U)}_{\text{locally cnst fns. on } \mathbb{R}^n \setminus A} \oplus \underbrace{H^0(V)}_{\text{locally cnst fns on } \mathbb{R}^n \setminus B} \text{ modulo cnst fns on } \mathbb{R}^n$$

It follow that if neither A nor B separates p from q, then neither does $A \cup B$. ∎

$\boxed{\text{NT}-6.3}$ Proof by induction on k.

- $\underline{\underline{k=0}}$ $$H^d(\mathbb{R}^n) = \begin{cases} 0 & d>0 \\ \mathbb{R} & d=0 \end{cases}$$ and result is confirmed

- $\underline{\underline{k=1}}$ $$H^d(\mathbb{R}^n - \{p\}) \simeq H^d(S^{n-1}) = \begin{cases} \mathbb{R} & d=0,n \\ 0 & \text{otherwise} \end{cases}$$ and the result is confirmed.

- Assume the result holds for k; and prove it holds for $k+1$. Let $\{p_0, p_1, \dots, p_k\}$ be $(k+1)$ distinct points in $\mathbb{R}^n$. Set $U = \mathbb{R}^n \setminus \{p_0\}$ and $V = \mathbb{R}^n \setminus \{p_1, \dots, p_k\}$

Then
- $U \cup V = \mathbb{R}^n \Rightarrow H^p(U \cup V) = \begin{cases} \mathbb{R} & p=0 \\ 0 & p>0 \end{cases}$

- $U \cap V = \mathbb{R}^n \setminus \{p_0, \dots, p_k\}$ Goal: Compute $H^p(U \cap V)$.

- By asm $H^d(V) = \begin{cases} \mathbb{R} & d=0 \\ \mathbb{R}^k & d=n-1 \\ 0 & \text{otherwise} \end{cases}$

- $n \geq 2 \Rightarrow U \cap V$ is connected $\Rightarrow \left(H^0(U \cap V) = \mathbb{R} \right)$

Mayer-Vietoris $\Rightarrow$ $0 \to H^p(U) \oplus H^p(V) \to H^p(U \cap V) \to 0$ is exact.

$\forall\ p>0$.
$$\Rightarrow H^p(U) \oplus H^p(V) \simeq H^p(U \cap V)$$

Result follows. ∎