Q1. Let \( f(x) = x^3 \). Using each of the three equivalences of Theorem 0.6, prove that this function is continuous.

Q2. Let \( E \) be a closed subset of \( \mathbb{R} \). Define \( f(x) \) by

\[
f(x) = \inf_{y \in E} |x - y|.
\]

Prove that \( f(x) \) is continuous and that \( f(x) = 0 \) if and only if \( x \in E \).

Q3. Let \( U \) be an open subset of \( \mathbb{R} \). For each \( x \in U \), prove that there is a largest open interval containing \( x \) and contained in \( U \). Call it \( J_x \), and show that any pair \( J_x, J_y \) are either equal or disjoint. Now show that \( U \) is a countable disjoint union of open intervals.

Q4. If \( E \subseteq \mathbb{R} \) is not compact, show that there is an unbounded continuous function on \( E \).

Q5. Prove that if \( X \) and \( Y \) are countable sets then

\[
X \times Y = \{(x, y) : x \in X, y \in Y\}
\]

is countable.

Q6. Let \( X \) and \( Y \) be disjoint subsets of a set \( S \) with \( X \) countable and \( Y \) infinite. Prove that \( X \cup Y \) has the same cardinality as \( Y \).

Q7. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. For each \( x \in \mathbb{R} \), define

\[
\omega(x) = \inf_{\delta > 0} \sup\{|f(y) - f(z)| : |x - y|, |x - z| < \delta\}.
\]

Prove that \( f \) is continuous at \( x \) if and only if \( \omega(x) = 0 \), and that \( \{x : \omega(x) < \varepsilon\} \) is open for all \( \varepsilon > 0 \) and for all \( x \in X \).