

MATH 446, HOMEWORK 7, DUE DEC 1
Everyone does Q1-Q5, honors students also do Q6, Q7

In Q1-Q4, the function $f(x)$ is always bounded.

Q1. If $f(x)$ is Riemann integrable on $[\varepsilon, 1]$ for $0 < \varepsilon < 1$, prove that is Riemann integrable on $[0, 1]$ and that

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 f(x) dx = \int_0^1 f(x) dx.$$

Q2. If $a < c < d < b$ and $f(x)$ is Riemann integrable on $[a, b]$, prove that it is Riemann integrable on $[c, d]$.

Q3. If $f(x)$ is Riemann integrable on $[a, c]$ and on $[c, b]$ for $a < c < b$, prove that it is Riemann integrable on $[a, b]$ and that

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Q4. If $f(x)$ is Riemann integrable on $[0, 1]$, prove that $e^{f(x)}$ is also Riemann integrable on $[0, 1]$. (MVT for e^t could be useful)

Q5. Let $T : C[0, 1] \rightarrow C[0, 1]$ be defined by

$$Tf(x) = \int_0^x f(t) dt, \quad x \in [0, 1], \quad f \in C[0, 1].$$

Prove that T is not a contraction (consider constant functions) but that T^2 is a contraction. Find the fixed point for T .

Q6. Let $f : (0, 1] \rightarrow (0, 1]$ be a contraction. Prove that f extends to a contraction $g : [0, 1] \rightarrow [0, 1]$.

Q7. Let U be an open connected set in \mathbb{R}^2 . Let $x_0 \in U$ be fixed, and let V be the set of points in U which can be connected to x_0 by a path in U . Prove that V and $V^c \cap U$ are both open sets. Deduce that U is path connected.