In Q1-Q4, the function \( f(x) \) is always bounded.

Q1. If \( f(x) \) is Riemann integrable on \([\varepsilon, 1]\) for \( 0 < \varepsilon < 1 \), prove that is Riemann integrable on \([0, 1]\) and that
\[
\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} f(x) \, dx = \int_{0}^{1} f(x) \, dx.
\]

Q2. If \( a < c < d < b \) and \( f(x) \) is Riemann integrable on \([a, b]\), prove that it is Riemann integrable on \([c, d]\).

Q3. If \( f(x) \) is Riemann integrable on \([a, c]\) and on \([c, b]\) for \( a < c < b \), prove that it is Riemann integrable on \([a, b]\) and that
\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx.
\]

Q4. If \( f(x) \) is Riemann integrable on \([0, 1]\), prove that \( e^{f(x)} \) is also Riemann integrable on \([0, 1]\). (MVT for \( e^t \) could be useful)

Q5. Let \( T : C[0, 1] \to C[0, 1] \) be defined by
\[
Tf(x) = \int_{0}^{x} f(t) \, dt, \quad x \in [0, 1], \quad f \in C[0, 1].
\]
Prove that \( T \) is not a contraction (consider constant functions) but that \( T^2 \) is a contraction. Find the fixed point for \( T \).

Q6. Let \( f : (0, 1) \to (0, 1) \) be a contraction. Prove that \( f \) extends to a contraction \( g : [0, 1] \to [0, 1] \).

Q7. Let \( U \) be an open connected set in \( \mathbb{R}^2 \). Let \( x_0 \in U \) be fixed, and let \( V \) be the set of points in \( U \) which can be connected to \( x_0 \) by a path in \( U \). Prove that \( V \) and \( V^c \cap U \) are both open sets. Deduce that \( U \) is path connected.