MATH607 THIRD HOMEWORK

Hand in the first three problems

Q1. Let $X$ be a set and let $\{X_i\}_{i \geq 1}$ be a disjoint collection of subsets whose union is $X$. On each $X_i$ let a $\sigma$–algebra $\mathcal{M}_i$ and a measure $\mu_i$ be given. Prove that

$$\mathcal{M} = \{ E \subseteq X : E \cap X_i \in \mathcal{M}_i \text{ for all } i \geq 1 \}$$

is a $\sigma$–algebra and that

$$\mu(E) = \sum_{i=1}^{\infty} \mu_i(E \cap X_i)$$

defines a measure on $\mathcal{M}$.

Q2. There are four natural ways to define the Borel subsets of $\mathbb{R}^2$: as the smallest $\sigma$–algebra generated by (a) the open sets, (b) the open balls, (c) the open rectangles, (d) all sets of the form $A \times B$ where $A$ and $B$ are Borel sets in $\mathbb{R}$. Prove that they are equivalent.

Q3. If $E$ is a Borel set in the plane as defined in Q2, prove that

$$E_y = \{ x : (x, y) \in E \}$$

is a Borel subset of $\mathbb{R}$ for all $y \in \mathbb{R}$.

Q4. Let $\lambda$ be Lebesgue measure on $\mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Prove that

$$\mu(E) = \lambda(f(E))$$

defines a measure on the Borel subsets of $\mathbb{R}$.

Q5. Let $\lambda$ be Lebesgue measure on the Lebesgue sets $\mathcal{L}$ in $\mathbb{R}$. Define a collection $\mathcal{M}$ of subsets of $\mathbb{R}$ by $E \in \mathcal{M}$ if, for each $\epsilon > 0$, there exists $F \in \mathcal{L}$ with $\lambda(F) < \epsilon$ and a continuous function $f$ such that $f = \chi_E$ on $F^c$. Prove that open intervals lie in $\mathcal{M}$, that $\mathcal{M}$ is closed under complements and finite intersections, and that $\mathcal{M}$ is a $\sigma$–algebra containing the Borel sets and $\mathcal{L}$. (I list these things as a hint on how to approach the problem)