Q1. p39 Q29

Q2. Let $S \subseteq [0, 1]$ be a dense set containing 0. and let $f : S \to [0, 1]$ be a function such that $f(S)$ is dense in $[0, 1]$. Suppose that if $x, y \in S$ and $x \leq y$, then $f(x) \leq f(y)$. Prove that

$$g(x) = \sup \{ f(z) : 0 \leq z \leq x, \ z \in S}\]$$

defines a continuous increasing function on $[0, 1]$ whose restriction to $S$ is $f$, and that $g$ is unique with respect to these properties. Complete the construction of the Cantor function.

Q3. p48 Q9a,b,c

Q4. Let $\mathcal{L}$ be the Lebesgue sets on $\mathbb{R}$, let $\lambda$ be Lebesgue measure, and let $\mu$ be a translation invariant measure on $\mathcal{L}$ such that $\mu([0, 1]) = 1$. The following steps will prove that $\mu = \lambda$.

(i) $\mu(\{x\}) = 0$ for any point.
(ii) $[a, b]$, $[a, b)$, $(a, b]$, $(a, b)$ all have the same $\mu$–measure.
(iii) $\mu([0, 1/n]) = 1/n$ for $n \in \mathbb{N}$.
(iv) $\mu([a, b]) = b - a$.
(v) $\mu(E) \leq \lambda(E)$ for $E \in \mathcal{L}$.
(vi) If $E \in \mathcal{L}$ and $E \subseteq [a, b]$ then $\mu(E) = \lambda(E)$.
(vii) $\mu(E) = \lambda(E)$ for $E \in \mathcal{L}$.

Q5. Let $(X, \Sigma)$ be a measurable space and let $\{f_n : X \to \mathbb{R}\}_{n \geq 1}$ be a sequence of measurable functions. Let

$$E = \{x \in X : \{f_n(x)\}_{n \geq 1} \text{ is a Cauchy sequence}\}.$$ 

Prove that $E \in \Sigma$. 