Solutions #4

81. (a) The translates of $N$ fill up $[0,1]$ and are disjoint so the translates of $E$ are disjoint and are contained in $[0,1]$. Thus $\mu(E) = 0$ otherwise $\mu([0,1]) = \infty$.

(b) If $m(E) > 0$ then $m(E \cap [n,n+1]) \neq 0$ for some $n$, so by translation we may assume that $E \subseteq [0,1]$. Since $E = \bigcup E \cap N_r$ and all $E \cap N_r$ are Lebesgue measurable, then $m(E \cap N_r) = 0$ by (a) so $m(E) = 0$ contradiction. Thus at least one $E \cap N_r$ is non-measurable.
Q2. It is immediate from the definition that
g is an increasing function. If there is a
point of discontinuity $x_0$ for $g$, then
\[ \lim_{x \to x_0^-} g(x) < \lim_{x \to x_0^+} g(x). \]

Call these $a$ and $b$. Then $g(x)$ does not take
any value in $(a, b)$, contradicting the density of
$f(s)$ in $[0, 1]$. Thus $g$ is continuous.

Suppose that $h$ is another such continuous
increasing function. For any point $x \in [0, 1]$ we
may choose points $x_n \in S$ such that $x_n \to x$, using
density of $S$. Then
\[ g(x) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} h(x_n) = h(x), \]
so $g = h$ proving uniqueness.
The initial step in defining the Cantor function is to define \( f \) on the removed intervals and for every \( n \) we get values:

\[
0, \quad \frac{j}{2^n} \text{ for } 1 \leq j < 2^n, \quad 1.
\]

This \( f \) has dense range to one part 1 to extend \( f \) to an increasingly continuous function.

Q3

a) \( g(x) = f(x) + x \).

\( f \) is increasing so \( g(x) \) is strictly increasing.

Thus, \( g \) maps \([0,1]\) to \([0,2]\). \( h = g^{-1} \) is increasing and onto so is continuous as in Q2.

b) \( g \) maps each middle \( \frac{1}{3} \) to an interval of the same length; a total of 1, so

\[
m(g(C)) = 2 \cdot 1 = 1.
\]

(c) \( g(C) \) contains a non-measurable set \( A \). Part
\[ B = g^{-1}(A). \] Then \( B \in \text{Cantor set} \Rightarrow \) is a null set and thus is in \( L \). To see that \( B \) is not Borel, we show that \( g \) maps Borel sets to Borel sets. Then if \( B \) is Borel we get \( A = g(B) \) is Borel, a contradiction.

Let \( M = \{ E : g(E) \text{ is Borel} \} \).

Since \( g([a,b]) = [g(a), g(b)] \), closed intervals are in \( M \). If \( E \in M \) and \( F \) is its complement in \( [0,1] \) then \( g(F) \) is the complement of \( g(E) \) in \( [0,1] \) so \( M \) is closed under complements.

\( M \) is closed under countable unions since

\[ g(\bigcup E_n) = \bigcup g(E_n), \] so \( M \) is a \( \sigma \)-algebra.

and so contains the Borel sets.
(i) \( \mu \left( \left[ 0, \frac{1}{n} \right] \right) \leq \frac{1}{n} \) since \( n \) disjoint translates of \( \left[ 0, \frac{1}{n} \right] \) fit in \( [0, 1] \). Thus

\[ \mu \left( \left[ 0, \frac{1}{n} \right] \right) \leq \frac{1}{n} \] for all \( n \to \infty \)

\[ \mu \left\{ 0 \right\} = 0, \text{ and } \mu \left\{ \frac{1}{n} \right\} = 0 \text{ by translation.} \]

(ii) \( \mu \left[ a, b \right] = \mu \left( a, b \right) + \mu \left[ a + \frac{1}{n}, b + \frac{1}{n} \right] = \mu \left( a, b \right) \)

so there four intervals have the same \( \mu \) measure.

(iii) \[ 1 = \mu \left[ 0, 1 \right] = \mu \left[ 0, \frac{1}{n} \right] + \mu \left[ \frac{1}{n}, \frac{2}{n} \right] + \ldots + \mu \left[ \frac{n-1}{n}, 1 \right] = n \mu \left[ 0, \frac{1}{n} \right]. \]

From (iii) \( \mu \left[ 0, \frac{1}{n} \right] = \mu \left( 0, \frac{1}{n} \right) = \frac{1}{n} \).

(iv) Look at \( [0, c] \). For any integer \( n \), choose

\( m \) such that \( \frac{m}{n} \leq c < \frac{m+1}{n} \). Then, by translates,

we have from (iii) that

\[ \frac{m}{n} \leq \mu \left[ 0, c \right] \leq \frac{m+1}{n}. \]

Thus \( c - \frac{m}{n} \leq \frac{1}{n} \) and \( \mu \left[ 0, c \right] - \frac{m}{n} \leq \frac{1}{n} \).
\( |c - \mu([0,c])| \leq 2/n \) for all \( n \geq 1 \)

so \( \mu([0,c]) = c \). Thus

\[
\mu([a,b]) = \mu([0,b-a]) = b-a.
\]

(v) This is clear if \( \lambda(E) = \infty \). If \( \lambda(E) < \infty \),
given \( \varepsilon > 0 \), choose open intervals \( (I_n)_n \), so that

\[
E \subseteq U I_n \quad \text{and} \quad \sum \lambda(I_n) < \lambda(E) + \varepsilon.
\]

Then \( \mu(E) \leq \sum \mu(I_n) = \sum \lambda(I_n) < \lambda(E) + \varepsilon. \)

Thus \( \mu(E) \leq \lambda(E) \) for all \( E \in \mathcal{L} \).

(vi) Let \( E \subseteq [a,b] \) and let \( F = E \cap [a,b] \).

Then \( \mu(E) + \mu(F) = b-a \) and \( \lambda(E) + \lambda(F) = b-a \).

If \( \mu(E) < \lambda(E) \) then \( \mu(F) > \lambda(E) \) contradicting

\[
(\ast). \quad \therefore \quad \mu(E) = \lambda(E)
\]

(vii) \( \mu(E) = \sum \mu(E \cap [n/n]) = \sum \lambda(E \cap [n/n]) \) (by vi)

\[ = \lambda(E). \]
For each \( n \), there exists \( m \) such that
\[
|f_i(x) - f_j(x)| < \frac{1}{n} \quad \text{for} \quad i, j \geq m.
\]

Let 
\[
E_{i,j,m} = \left\{ x : f_i(x) - f_j(x) \in (-\frac{1}{n}, \frac{1}{n}) \right\}.
\]

Put 
\[
F_{m,n} = \bigcap_{i,j \geq m} E_{i,j,m}.
\]

\[
G_n = \bigcup_{m=1}^{\infty} F_{m,n}.
\]

\[
H = \bigcap_{n \geq 1} G_n.
\]

\( H \) is a measurable net and
\( H \) is precisely where \( \{f_n(x)\} \) is Cauchy.