ON THE COHOMOLOGY GROUPS OF CERTAIN VON NEUMANN ALGEBRAS WITH COEFFICIENTS IN K(H)

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ABSTRACT. We prove that the normal cohomology groups $H_w^n(M, K(H))$ of a von Neumann algebra M with coefficients in the algebra of compact operators are zero if M is atomic of type $\mathrm{I}_{\mathrm{fin}}$. In addition, the completely bounded normal cohomology groups $H_{wcb}^n(B(H), K(H))$ are shown to be 0 as well.

1. INTRODUCTION

The study of the Hochschild cohomology theory for von Neumann algebras was initiated in the early 70's in the pioneering work of Kadison, Johnson, and Ringrose [8, 9, 5]. Since then the theory has seen significant progress and, while nowhere near completion, it is reasonable to say that it has reached maturity. It is useful at this point to briefly recall the main definitions.

If M is a von Neumann algebra and X is a Banach M-bimodule, let $L^k(M, X)$ be the Banach space of k-linear bounded maps from M^k into X, $k \geq 1$. For $k = 0$, $L^0(M, X)$ is taken to be X. The coboundary operator $\partial^k : L^k(M, X) \to L^{k+1}(M, X)$ (usually abbreviated to just ∂) is defined, for $k \geq 1$, by

$$\begin{aligned}\partial\varphi(x_1, \ldots, x_{k+1}) =& x_1\varphi(x_2, \ldots, x_{k+1}) \\ &+ \sum_{i=1}^{k} (-1)^i \varphi(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_{k+1}) \\ &+ (-1)^{k+1}\varphi(x_1, \ldots, x_k)x_{k+1},\end{aligned}$$

for $x_1, \ldots, x_{k+1} \in M$. When $k = 0$, we define $\partial\xi$, for $\xi \in X$, by

$$\partial\xi(x) = x\xi - \xi x, \qquad x \in M$$

The coboundary operator satisfies $\partial^{k+1}\partial^k = 0$, and, consequently, Im ∂^k (the space of coboundaries) is a subspace of Ker ∂^{k+1} (the space of cocycles). The continuous Hochschild cohomology groups $H^k(M, X)$ are then defined to be the quotient vector spaces Ker ∂^k/Im ∂^{k-1}, $k \geq 1$.

Since M is the dual of M_*, the wealth of topological and measure theoretical properties of M has led, from the beginning, to the additional assumption that cocycles and coboundaries are normal (i.e. separately ultraweakly continuous in each variable). This, in turn, required that the M-modules X be duals of

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Banach spaces themselves. The cohomology groups under these circumstances are denoted by $H_w^k(M,X)$, and it was proved [5] that $H_w^k(M,X) = H^k(M,X)$ for all von Neumann algebras M and all dual M-modules X.

The most relevant cases turned out to be $X = M$ and $M = B(H)$, and the Kadison-Johnson-Ringrose conjecture, stating that $H^k(M,X) = 0$, has been verified for large classes of von Neumann algebras. We mention here the types I [8], II_∞ and III [2], as well as several classes of type II_1 factors, such as those with property Γ [4]. Finite von Neumann algebras with Cartan subalgebras were proved to satisfy $H^k(M,M) = 0$ for all k [11, 3, 16].

It is important to point out that the vanishing of $H^1(M,X)$ means that all derivations $\delta : M \to X$ (i.e. $\delta(xy) = \delta(x)y + x\delta(y)$) are inner, that is, $\delta(x) = ax - xa$ for some $a \in X$. The case $n = 1$ and $X = M$ was, historically, the first to be settled ([7],[14]), and now there exist fairly short proofs for it. By contrast, the case $X = B(H)$ is still open and it was proved by Kirchberg [10] to be equivalent to Kadison's similarity problem. The latter asks if every bounded representation of a C*-algebra is similar to a *-representation.

The monograph [15], together with the updated paper [17], survey the current state of knowledge in this area.

The case when the M-bimodule X is not a dual space has known a very different fate, by comparison. The showcase example is, of course, the algebra $K(H)$ of compact operators on a Hilbert space H, but even here major difficulties occurred from the very beginning. It is not surprising that derivations were studied first. Johnson and Parrott [6] approached the issue in the early 70's and had to develop new and deep techniques to prove the vanishing of $H^1(M, K(H))$. They left open the type II_1 case, which was answered in the affirmative by Popa [12] some fifteen years later. The passage to $n \geq 2$ turned out to be more difficult than expected, and there is only one paper to date that has dealt with the problem. In [13], Rădulescu proved the vanishing of $H_w^2(M, K(H))$ for type II_1 factors which have both property Γ and a Cartan subalgebra.

In this paper we investigate the vanishing of $H_w^n(M, K(H))$ and we obtain some positive answers for type I von Neumann algebras. In section 2 of this paper we prove that $H_w^n(M, K(H)) = 0$ for all $n \geq 1$ if M is an atomic von Neumann algebra of type $\mathrm{I}_{\mathrm{fin}}$, that is, a countable direct sum of tensor products of matrix algebras and discrete abelian algebras. In section 3 we show that the completely bounded normal cohomology group $H_{wcb}^n(B(H), K(H))$ is also 0. The first section contains a collection of technical results on the so^*-to-norm continuity of $K(H)$-valued linear and multilinear maps.

2. CONTINUITY PROPERTIES OF NORMAL MAPS INTO $K(H)$

In this section we address the issue of continuity of linear and multilinear maps on von Neumann algebras with values in $K(H)$. Throughout this section, normality of these maps is understood when viewing them as taking values in $B(H)$. All Hilbert spaces are assumed to be separable, and, unless otherwise specified, infinite-dimensional. We begin with a well-known fact which we state as a remark:

Remark 2.1. The strong*-operator (from now on referred to as so^*) topology on the unit ball of $B(H)$ is metrizable. Indeed, take (ξ_n) to be a countable dense set of vectors in the unit ball of H and define $||x||_0^2 = \sum 2^{-n}\ ||x\xi_n||^2 + \sum 2^{-n}\ ||x^*\xi_n||^2$. Then $\mathrm{so}^* - \lim x_\alpha = 0$ if and only if $\lim ||x_\alpha||_0 = 0$. □

This observation allows us to use sequences, rather than nets, to prove so^*-continuity of maps on the unit ball of $B(H)$. The next result is due to F. Rădulescu:

Proposition 2.2 ([13]). *Let $\varphi : \ell^\infty \to K(H)$ be a bounded, normal map. Then φ is continuous from the unit ball of ℓ^∞ with the weak-operator topology to $K(H)$ with the norm topology.*

Theorems about strong-to-norm continuity of $K(H)$–valued normal maps originate in [12]. Extending a result in [12] for derivations, it was proved in [13] that if $\varphi : M \to K(H)$ is a bounded, normal map on a finite von Neumann algebra, then φ is continuous from the unit ball of M with the so^* topology to $K(H)$ with the norm topology. Our next result belongs to this circle of ideas.

Proposition 2.3. *Let $M \subseteq B(H)$ be a von Neumann algebra which is isomorphic to $B(H)$. If $\varphi : M \to K(H)$ is a bounded normal map, then φ is continuous from the unit ball of M with the so^* topology to $K(H)$ with the norm topology.*

Proof. Since isomorphisms of von Neumann algebras preserve the so^* topologies on the unit balls, we may assume that $M = B(H)$. Let (x_n) be a bounded sequence convergent so^* to 0. We may clearly assume that $x_n^* = x_n$, and consider the polar decomposition $x_n = u_n|x_n|$, where u_n are unitary operators. It follows that $|x_n| = u_n^* x_n$ is so^* convergent to 0. Since $(x_n)_+ + (x_n)_-$ and $(x_n)_+ - (x_n)_-$ both converge so^* to 0, it suffices to study the case when the x_n's are positive. Assume first that $x_n = p_n$ are projections, and that there exists $c > 0$ such that, by passing to a subsequence if necessary, we have $||\varphi(p_n)|| > 2c$. We will construct by induction a sequence (e_n) of mutually orthogonal, finite dimensional projections such that $||\varphi(e_n)|| > c$. Since $||\varphi(p_1)|| > c$ and p_1 is a weak limit of finite-dimensional sub-projections, lower semi-continuity of the norm ensures the existence of a finite dimensional e_1 such that $||\varphi(e_1)|| > c$. Suppose that mutually orthogonal, finite dimensional projections $e_1, \ldots, e_n$ have been constructed such that $||\varphi(e_i)|| > c$ for all $i = 1, 2, \ldots, n$. Fix $0 < \varepsilon < c/2||\phi||$, let $e = e_1 + \ldots + e_n$, and note that $||ep_n||$ converges to 0, hence $||(I-e)p_n(I-e) - p_n|| < \varepsilon$ for large enough n. Then the spectrum of $(I-e)p_n(I-e)$ in $B((I-e)H)$ is included in the spectrum of $(I-e)p_n(I-e)$ in $B(H)$, and the latter is a subset of $[-\varepsilon,\varepsilon]\cup[1-\varepsilon,1+\varepsilon]$. By functional calculus, there exists a projection f in $B((I-e)H)$ such that $||(I-e)p_n(I-e)-f||<\varepsilon$, hence $||f-p_n||<2\varepsilon$. Since $||\varphi(p_n)||>2c$, we have $||\varphi(f)||>||\varphi(p_n)||-||\varphi(f-p_n)||>c$. Now approximate f well enough by a finite dimensional projection $e_{n+1}\leq f$ in $B((I-e)H)$ still satisfying $||\varphi(e_{n+1})||>c$. This completes the inductive step. The von Neumann algebra generated by the projections e_n is isomorphic to ℓ^∞ and the restriction of φ to this algebra satisfies $||\varphi(e_n)||>c$, contradicting Proposition 1.2. It follows that $\lim||\varphi(p_n)||=0$.

If $0\leq x_n\leq I$, then let $x_n=\sum_m 2^{-m}f_m^n$ be the diadic decomposition of x_n. Then $(f_m^n)_n$ converges strongly to 0 for every fixed m (for instance, f_1^n is defined by $x_nf_1^n\geq 1/2f_1^n,\ x_n(I-f_1^n)\leq 1/2f_1^n$). Fix $\varepsilon>0$ and m_0 such that $2^{-m_0}<\varepsilon/2||\varphi||$. Then, by the first part of the proof, there exists n_0 such that for $n\geq n_0,\ \varphi(f_m^n)<\varepsilon/2$ for any $m\leq m_0$. Thus, for $n\geq n_0$,

$$||\varphi(x_n)||\leq\sum_{m=1}^{m_0}2^{-m}||\varphi(f_m^n)||+||\varphi||\sum_{m>m_0}2^{-m}\leq\varepsilon/2+\varepsilon/2\leq\varepsilon.$$

□

Remark 2.4. The above result also holds true if M is the range of a normal conditional expectation $\Phi:B(H)\to M$. If $\varphi:M\to K(H)$ is a bounded, normal map, then $\varphi\circ\Phi$ is normal from $B(H)$ with values in $K(H)$, hence continuous from the unit ball of $B(H)$ with the so^* topology to $K(H)$ with the norm topology. Since φ is the restriction of $\varphi\circ\Phi$ to M, it inherits the same continuity property. □

Proposition 2.5. *Let $M_1,\ldots,M_k$ be von Neumann algebras with unit balls $B_1,\ldots,B_k$, and let Z be a Banach space with the norm topology. Let $\varphi\colon M_1\times\ldots\times M_k\to Z$ be a bounded multilinear map, separately so^*-to-norm continuous in each variable. Then φ is jointly continuous on $B_1\times\ldots\times B_k$ for these topologies.*

Proof. The argument is essentially the same as 4.1, 4.2 and 4.3 of [4], using the fact that each B_i is a complete metric space in the so^* topology. However, there are two differences which must be discussed. As in [4], it suffices to consider the initial case $k=2$ and then proceed by induction on k.

In [4], each M_i was type II_1 and we had a self-adjoint sequence $\{h_n\}_{n=1}^\infty$ from B_1 converging to 0 in $\|\cdot\|_2$–norm, from which $\lim_{n\to\infty}\|h_n^\pm\|_2=0$ was deduced from the equality

$$\|h_n\|_2^2=\|h_n^+\|_2^2+\|h_n^-\|_2^2.$$

This is now replaced by a self-adjoint sequence $\{h_n\}_{n=1}^\infty$ converging to 0 in the so^* topology. Since h_n^+ is obtained from h_n by applying the functional calculus to the function $g(t)=t\vee 0$ on $[-1,1]$, so^* convergence of $\{h_n^+\}_{n=1}^\infty$ to 0 then follows from [19, Lemma II.4.3], with a similar argument for $\{h_n^-\}_{n=1}^\infty$.

The second difference is more substantial. In [4], joint continuity of φ at a point $(a,0)$ was obtained and then used to prove joint continuity at $(0,0)$ via the polar decomposition $a = bu$ with $b \geq 0$. In the type II_1 setting of [4], the argument relied on being able to take u to be a unitary. In the present context, this is only possible when a is self-adjoint, in which case joint continuity at $(0,0)$ follows as in [4]. To handle the general case, let $M_0 = \mathbb{M}_2(M_1)$, let $A \in M_0$ be the self-adjoint matrix $\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ and define $\psi \colon M_0 \times M_2 \to Z$ by

$$\psi(X,y) = \varphi(x_{12}, y), \quad X = (x_{ij}) \in M_0, \quad y \in M_2.$$

Then ψ is jointly continuous at $(A,0)$ on $B_0 \times B_2$, and hence at $(0,0)$ from above. This then implies joint continuity of φ at $(0,0)$.

With these two modifications, the proof is now exactly as in [4], although the 2×2 matrix argument is also used in the inductive step. $\square$

Corollary 2.6. *Let $M \subseteq B(H)$ be a von Neumann algebra which is isomorphic to $B(H)$ and let B denote the unit ball. Let $\varphi \colon M \times \ldots \times M \to K(H)$ be a bounded multilinear map, separately normal in each variable. Then φ is jointly continuous on $B \times \ldots \times B$, where each copy of B has the so* topology and $K(H)$ has the norm topology.*

Proof. This is an immediate consequence of Propositions 2.3 and 2.5; in the latter, take the Banach space Z to be $K(H)$. $\square$

The next result is probably folklore, but we could not find an appropriate reference for it (see, however, [1, Cor. III.10]). Since it will be important to us in the next section, we state it and include a sketch of proof.

Proposition 2.7. *Let M denote a von Neumann algebra and let $\varphi \colon M \to \mathbb{C}$ be a bounded linear functional. Then φ is normal if and only if φ is continuous on the unit ball of M with the so* topology.*

Proof. If φ is normal, it is obviously strongly* continuous, so assume that φ is so^* continuous on the unit ball of M.
Since $\varphi(x) = \dfrac{(\varphi(x) + \overline{\varphi(x^*)})}{2} + i\dfrac{(\varphi(x) - \overline{\varphi(x^*)})}{2i}$, we may assume that φ is self-adjoint. To prove that φ is normal, it suffices to show that kerφ is ultraweakly closed. By the Krein-Smulian theorem, it suffices to show that $B \cap \ker\varphi$ is ultraweakly closed, where B is the unit ball of M. Let (x_α) be a net in $B \cap \ker\varphi$, convergent ultraweakly to x. Then $x_\alpha^* \in B \cap \ker\varphi$, and, since $x_\alpha = \dfrac{(x_\alpha + x_\alpha^*)}{2} + i\dfrac{(x_\alpha - x_\alpha^*)}{2i}$, we may assume that $x_\alpha = x_\alpha^*$. There exists a net (y_β) of convex combinations of the (x_α)'s which is strongly (hence so^*) convergent to x. By hypothesis, $\varphi(x) = \lim \varphi(y_\beta) = 0$, so $x \in \ker\varphi$. $\square$

The analogue of Corollary 2.6 in this context is

Proposition 2.8. *Let $M \subseteq B(H)$ be a von Neumann algebra, isomorphic to $B(H)$, and with unit ball B. Let $\varphi\colon M \times \ldots \times M \to \mathbb{C}$ be a bounded multilinear functional, separately normal in each variable. Then φ is jointly so* continuous on $B \times B \times \ldots \times B$.*

We conclude this section with an application of continuity which highlights the structure of $K(H)$–valued maps.

Proposition 2.9. *Let M be a von Neumann algebra with an increasing sequence of projections (p_n) convergent to I. If $\varphi\colon M \to K(H)$ is a bounded normal map, then*

$$\lim_n \ ||\varphi|_{(I-p_n)M(I-p_n)}|| = 0.$$

Proof. To reach a contradiction, suppose that, by passing if necessary to a subsequence, there exists $c > 0$ and $x_n \in M$ of norm $||x_n|| \leq 1$ such that $||\varphi((I-p_n)x_n(I-p_n))|| > 4c$ for all $n \geq 1$. Since the sequence p_n increases to I and φ is normal, ultraweak lower semi-continuity of the norm ensures the existence of an integer n_1 such that $||\varphi((p_{n_1}-p_1)x_1(p_{n_1}-p_1))|| > 4c$. Since every contraction can be written as $a_1 - a_2 + i(b_1 - b_2)$ for positive contractions a_1, a_2, b_1, b_2, we obtain a positive contraction $y_1 \in (p_{n_1}-p_1)M(p_{n_1}-p_1) \subset p_{n_1}Mp_{n_1}$ such that $||\varphi(y_1)|| > c$. The diadic decomposition of y_1 ensures the existence of a projection $e_1 \in p_{n_1}Mp_{n_1}$ satisfying $||\varphi(e_1)|| > c$. The argument is repeated after replacing M by $(I-p_{n_1})M(I-p_{n_1})$ and continues by induction to obtain a sequence (e_n) of mutually orthogonal projections in M such that $||\varphi(e_n)|| > c$, in contradiction to Proposition 1.2. □

3. BOUNDED COHOMOLOGY

Consider a von Neumann algebra $M = \bigoplus_i B(H_i)\overline{\otimes}A_i$, where all Hilbert spaces H_i are finite dimensional and all abelian algebras A_i are discrete (that is, they have minimal projections). Note that M is isomorphic to a block-diagonal subalgebra of $B(H)$, that is, to a subalgebra of the form $M = \bigoplus_j e_jB(H)e_j$, where (e_j) is a sequence of mutually orthogonal, finite-dimensional projections with $\sum_j e_j = I$. Then M is the range of a normal conditional expectation on $B(H)$, so the conclusion of Proposition 2.3 applies to M because of Remark 2.4. On the other hand, we make the key observation that any sequence of unitary operators in M contains a subsequence which converges so* to a unitary operator.

Let $\varphi\colon M^k \to K(H)$ be a separately normal k-cocycle. It is a well-known fact that we may assume without loss of generality that φ is 0 if any of its entries is scalar.

Since M is generated by an amenable group of unitaries G, define

$$\alpha(x_1, \ldots, x_{k-1}) = \int_G u^*\varphi(u, x_1, \ldots, x_{k-1})d\mu(u)$$

where μ is an invariant mean. It is obvious that α vanishes when any entry is scalar, hence the same is true for $\partial\alpha$.

Suppose that $\alpha(x_1, \dots, x_{k-1})$ is not compact. Then there exists a sequence of projections $q_n \in B(H)$ decreasing to 0, and $c > 0$, such that

$$||\alpha(x_1, \dots, x_{k-1})q_n|| > c \text{ for all } n \geq 1.$$

Since α is an ultraweak limit of convex combinations, the ultraweak inferior semicontinuity of the norm ensures the existence of a sequence (U_n) in G satisfying

$$||(U_n)^*\varphi(U_n, x_2, \dots, x_{k-1})q_n|| > c \Leftrightarrow ||\varphi(U_n, x_2, \dots, x_{k-1})q_n|| > c.$$

By passing, if necessary, to a subsequence, we have $\mathrm{so}^* - \lim U_n = U$, which implies

$$\begin{aligned} 0 < c &< ||\varphi(U_n, x_2, \dots, x_{k-1})q_n|| \\ &< ||\varphi(U_n - U, x_2, \dots, x_{k-1})q_n|| + ||\varphi(U, x_2, \dots, x_{k-1})q_n||. \end{aligned}$$

This is a contradiction, since $||\varphi(U_n - U, x_2, \dots, x_{k-1})||$ has limit 0 as a consequence of Proposition 2.3, while $||\varphi(U, x_2, \dots, x_{k-1})q_n||$ has limit 0 because $\varphi(U, x_2, \dots, x_{k-1})$ is a compact operator. This concludes the proof that $\alpha(x_1, \dots, x_{k-1}) \in K(H)$.

We will now show that α is separately normal in each variable. To keep notation simple, we prove this for the first variable, the argument being identical for each of the other variables. To this extent, fix $x_2, \dots, x_{k-1} \in M$ and an arbitrary normal functional θ on $B(H)$. All we have to prove is that the functional on M defined by $x \mapsto \theta \circ \alpha(x, x_2, \dots, x_{k-1})$ is normal. According to Proposition 2.7, this will be the case once we prove so^* continuity on B, the unit ball of M.

Let y_n be a sequence in B convergent so^* to 0. To get a contradiction, suppose that there exists $c > 0$ such that, by passing if necessary to a subsequence,

$$|\theta \circ \alpha(y_n, x_2, \dots, x_{k-1})| > c.$$

By arguing as in the first part of the proof, there are unitary operators U_n such that

$$|\theta((U_n)^*\varphi(U_n, y_n, x_2, \dots, x_{k-1}))| > c.$$

By passing, if necessary, to a subsequence, we have $\mathrm{so}^* - \lim U_n = U$, which implies

$$\begin{aligned} 0 < c &< |\theta((U_n)^*\varphi(U_n, y_n, x_2, \dots, x_{k-1}))| \\ &\leq |\theta((U_n - U)^*\varphi(U_n - U, y_n, x_2, \dots, x_{k-1}))| \\ &\quad + |\theta((U_n - U)^*\varphi(U, y_n, x_2, \dots, x_{k-1}))| \\ &\quad + |\theta(U^*\varphi(U_n - U, y_n, x_2, \dots, x_{k-1}))| \\ &\quad + |\theta(U^*\varphi(U, y_n, x_2, \dots, x_{k-1}))|. \end{aligned}$$

The last four terms have limit 0, the first three as a consequence of Proposition 2.8, while the fourth as a consequence of Proposition 2.7, which represents a contradiction. The conclusion we reach is that $\varphi - \partial\alpha$ is left- M-modular in the first variable. Combined with the fact that $\varphi - \partial\alpha$ is 0 when its first entry is scalar, we obtain that $\varphi - \partial\alpha = 0$.

This discussion proves the following:

Theorem 3.1. *If $M = \bigoplus_i B(H_i)\overline{\otimes}A_i$, where all Hilbert spaces H_i are finite-dimensional and all abelian algebras A_i are discrete, then $H_w^n(M, K(H)) = 0$ for all $k \geq 1$.*

4. COMPLETELY BOUNDED COHOMOLOGY

In this section we study completely bounded cohomology with values in $K(H)$. For this purpose, we begin with some remarks on the structure of completely bounded maps with values in $K(H)$.

Lemma 4.1. *If $M \subset B(H)$ is a von Neumann algebra, and $\varphi\colon M \to B(H)$ is a completely bounded, normal map, then there exist operators $a_1, \ldots, a_n, \ldots$ and $b_1, \ldots, b_n, \ldots$ in $B(H)$ such that $\varphi(x) = \sum_{i=1}^{\infty} a_i x b_i$ for all $x \in B(H)$, where the convergence is point-strong. Moreover, the row operator $A = (a_1, a_2, \ldots)$ and the column operator $B = (b_1, b_2, \ldots)^T$ are bounded.*

Proof. It is well known that $\varphi(x) = A_0\pi(x)B_0$, where π is a normal representation. Any normal representation has the form $\pi(x) = UP'(x \otimes I_{B(\ell^2)})U^*$ for some unitary U and some projection P' commuting with $M \otimes I_{B(\ell^2)}$ (see [18, E.8.8]), therefore $\varphi(x) = A_0UP'(x \otimes I_{B(\ell^2)})U^*B_0$. The fact that φ takes values in the same $B(H)$ forces $A = A_0UP'$ to be a bounded row operator and $B = U^*B_0$ to be a bounded column operator. The conclusion follows. □

Lemma 4.2. *There exists an amenable group of unitary operators $\mathcal{U} \subset B(H)$ such that $\mathcal{U}'' = B(H)$ and with the property that the map $\theta\colon B(H) \to B(H)$ defined by $\theta(x) = \int_{\mathcal{U}} u^*xu \, d\mu(u)$ vanishes on $K(H)$.*

Proof. Fix an orthonormal basis $\{\xi_i\}$ for H, and let p_n denote the projection onto the span of $\xi_1, \ldots, \xi_n$. Denote by $\mathcal{U}_n = \{u_n + (I - p_n),\ u_n \in p_nB(H)p_n \text{ unitary}\}$ and let $\mathcal{U} = \cup_n \mathcal{U}_n$. Denote by μ_n and μ left-invariant means on $\mathcal{U}_n$ and $\mathcal{U}$, respectively. Fix a compact operator K with the purpose of showing that $\theta(K) = 0$. Since θ is norm-to-norm continuous, we may assume that $K = p_nKp_n$ for some $n \geq 1$. We have

$$\theta(K) = \lim_m \int_{\mathcal{U}_m} u^*Ku \, d\mu_m(u) = \lim_m \frac{1}{m} tr(K)p_m = 0$$

where tr is the trace on $B(H)$. □

Proposition 4.3. *Let φ be a completely bounded, normal map from $B(H)$ to $K(H)$ and let $\mathcal{U} \subset B(H)$ be an amenable group of unitary operators as in Lemma 4.2. Then the map $T_\varphi\colon B(H) \to B(H)$ defined by*

$$T_\varphi(x) = \int_{\mathcal{U}} u^*\varphi(ux)\ d\mu(u)$$

is normal, satisfies $||T_\varphi|| \leq ||\varphi||$, and takes values in $K(H)$.

Proof. By writing $\varphi(x) = A_0\pi(x)B_0$, where π is a normal representation, we obtain $T_\varphi(x) = (\int_{\mathcal{U}}\ u^*A_0\pi(u)\ d\mu(u))\pi(x)B = A_1\pi(x)B$, which shows that T_φ is completely bounded and normal from $B(H)$ to $B(H)$. All that remains to be proved is that $T_\varphi(x) \in K(H)$.

Fix $\varepsilon > 0$, an orthonormal basis $\{\xi_i\}$ for H, and let p_n denote the projection onto the span of $\xi_1, \ldots, \xi_n$. By Proposition 2.9, choose a finite rank projection $p_n = p$ such that $||\varphi|_{(I-p)B(H)(I-p)}|| < \varepsilon$. This enables us to write

$$\begin{aligned}\varphi(x) &= \varphi(pxp) + \varphi(px(I-p)) + \varphi((I-p)xp) + \varphi((I-p)x(I-p)) \\ &= \varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x)\end{aligned}$$

If we represent $\varphi(x) = \sum_{i=1}^\infty\ a_ixb_i$ as in Lemma 4.1, then $\varphi_1(x) = \sum_{i=1}^\infty\ a_ipxpb_i$ and $T_{\varphi_1}(x) = \sum_{i=1}^\infty\ (\int_{\mathcal{U}} u^*a_ipu\ d\mu(u))xpb_i = 0$ since, by Lemma 4.2, the averages of a_ip are 0. A similar argument shows that $T_{\varphi_2}(x) = 0$.

We have $T_{\varphi_3}(x) = \sum_{i=1}^\infty\ (\int_{\mathcal{U}} u^*a_i(I-p)u\ d\mu(u))xpb_i = \sum_{i=1}^\infty\ \lambda_ixpb_i$, where λ_i are scalars and $(\lambda_1, \lambda_2, \ldots)$ is a bounded row. Then the series $\sum_{i=1}^\infty\ \lambda_ixpb_i$ is convergent in norm, hence it is an element of $K(H)$.

Since $||T_{\varphi_4}|| \leq ||\varphi_4|| < \varepsilon$, we get $||T_\varphi - T_{\varphi_3}|| \leq \varepsilon$ and $T_{\varphi_3}(x) \in K(H)$, therefore $T_\varphi(x) \in K(H)$. □

We are now in the position to prove the main result of this section.

Theorem 4.4. *For $n \geq 2$, $H^n_{wcb}(B(H), K(H)) = 0$.*

Proof. Let ψ be a completely bounded, separately normal n-cocycle, assumed to vanish whenever either of the variables is a scalar. Let $\mathcal{U} \subset B(H)$ be an amenable group of unitary operators like in Lemma 4.2. Since $\psi(x_1, \ldots, x_n) = A_1\pi_1(x_1)A_2\pi_2(x_2)A_3 \ldots A_n\pi_n(x_n)A_{n+1}$ for some normal representations $\pi_1, \ldots, \pi_n$, it is easy to see that

$$\alpha(x_2, \ldots, x_n) = \int_{\mathcal{U}} u^*\psi(u, x_2, \ldots, x_n)\ d\mu(u) = B\pi_2(x_2)A_3 \ldots A_n\pi_n(x_n)A_{n+1}$$

is completely bounded and separately normal in each variable, where $B = \int_{\mathcal{U}}\ u^*A_1\pi(u)A_2\ d\mu(u)$.

We will prove that α takes values in $K(H)$. For this purpose, fix $x_2, \ldots, x_n \in B(H)$ and define the completely bounded normal $\varphi\colon B(H) \to K(H)$ by $\varphi(x) = \psi(x, x_2, \ldots, x_n)$. According to Proposition 4.3, the map

$$\gamma(x) = \int_{\mathcal{U}} u^*\varphi(ux)\ d\mu(u)$$

takes values in $K(H)$. In particular, $\alpha(x_2, \ldots, x_n) = \gamma(I) \in K(H)$, which proves the claim. Finally, consider $\beta = \psi - \partial\alpha$. Then β is a completely bounded, separately normal, and $B(H)$-left-modular in the first variable. Since β is 0 when the first variable is a scalar, it follows that $\beta = 0$. □

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