# The spatial isomorphism problem for close separable nuclear C$^*$-algebras

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**The Kadison–Kastler problem asks whether close C$^*$-algebras on a Hilbert space must be spatially isomorphic. We establish this when one of the algebras is separable and nuclear. We also apply our methods to the study of near inclusions of C$^*$-algebras.**

Key Words: operator algebras | perturbations | nuclear C$^*$-algebras

## Introduction

Perturbation theory for operator algebras $A$ and $B$ on a Hilbert space $H$ was initiated in 1972 by Kadison and Kastler, [1]. They introduced a metric $d(\cdot,\cdot)$ on the set of norm closed subalgebras of the bounded operators $\mathbb{B}(H)$ on $H$, the simplest description of which is the following: $d(A,B)$ is the infimum of all positive constants $\lambda$ with the property that each element of the unit ball of one algebra is within a distance $\lambda$ of an element of the unit ball of the other algebra. These authors laid the groundwork for the subsequent theory, focusing on the important class of von Neumann algebras, those which are not only self-adjoint and norm closed (a C$^*$-algebra), but are also closed in the strong operator topology. An algebra $B$ that is close to an algebra $A$ is often referred to as a perturbation of $A$, and a major theme of [1] was to show that the Murray–von Neumann type classification of von Neumann algebras is preserved under small perturbations. Given an algebra $A \subseteq \mathbb{B}(H)$, there is one natural way to construct close algebras: take a unitary operator $u \in \mathbb{B}(H)$ that is close to the identity operator $I_H$ and form the algebra $B = uAu^*$. A simple calculation gives $d(A,B) \leq 2\|I_H - u\|$. Kadison and Kastler conjectured the converse statement, that close operator algebras should be unitarily equivalent. In this note we will describe our solution to this problem for the broad class of separable nuclear C$^*$-algebras, defined in the next section. Full details and proofs will appear elsewhere in a longer account.

A natural approach to the Kadison–Kastler problem is to break it into subproblems. In increasing order of strength, these are as follows:

(i) If $A$ and $B$ are close, then do they have shared properties?

(ii) If $A$ and $B$ are close, then must they be isomorphic?

(iii) If an isomorphism in (ii) can be found, then can it be chosen to be implemented by a unitary $u \in \mathbb{B}(H)$?

(iv) If a unitary $u \in \mathbb{B}(H)$ in (iii) can be found, then can it be chosen to be close to $I_H$?

Significant progress on all of these problems has been made by many authors, and we briefly describe some of the major developments that are particularly relevant to the present paper.

Soon after the publication of [1], E.C. took up the study of perturbation theory in a series of papers [2, 3, 4]. In [3], it was shown that the Kadison–Kastler problem has an affirmative answer for close pairs of injective von Neumann algebras $M$ and $N$, and the unitary $u \in \mathbb{B}(H)$ that implements the isomorphism can be chosen to satisfy the inequality

$$\|u - I_H\| \leq 19\, d(M,N)^{1/2},$$

and to lie in the von Neumann algebra $(M \cup N)''$ generated by $M$ and $N$. Subsequently, Raeburn and Taylor showed in [5] that injectivity is a shared property for close von Neumann algebras, so it is sufficient to only assume injectivity of one of the algebras above. In general, close operator algebras have many shared properties; two of most concern to us are norm separability and nuclearity, [4]. E.C. also introduced the useful and more general notion of a one-sided near containment $A \subset_\gamma B$. The defining property is that there should be a constant $\gamma' < \gamma$ so that each $x$ in the unit ball of $A$ has a corresponding element $y \in B$ satisfying $\|x - y\| \leq \gamma'$. The spatial isomorphism of close injective von Neumann algebras has an analog in the one-sided case: for an injective von Neumann $M$ and a near inclusion $M \subset_\gamma N$, there exists a unitary $u \in (M \cup N)''$ with $uMu^* \subseteq N$, again with an estimate on $\|u - I_H\|$ in terms of $\gamma$.

In [6], Johnson constructed pairs of separable nuclear C$^*$-algebras $(A_n, B_n)$, $n \geq 1$, with $\lim\limits_{n\to\infty} d(A_n, B_n) = 0$, and having special properties. Each is an isomorphic pair, even unitarily conjugate, but each isomorphism $\phi_n\colon\ A_n \to B_n$ is uniformly bounded away from the identity, and each unitary implementing an isomorphism is uniformly bounded away from $I_H$. These examples are the images of representations of $C[0,1] \otimes K(H)$ and so are nuclear C$^*$-algebras. In subsequent joint work with Choi, [7], E.C. constructed a sequence of arbitrarily close nonisomorphic pairs of C$^*$-algebras, for which nonseparability in the norm topology played an essential role. Together, these sets of examples provide counterexamples to (ii) for nonseparable nuclear C$^*$-algebras and to (iv) for separable nuclear C$^*$-algebras. As a consequence, (iii) is the right setting for the Kadison–Kastler problem in the context of separable C$^*$-algebras while (iv) may be true for all von Neumann algebras, although restricting to separably acting ones would avoid any potential pathologies arising from nonseparability. It should be noted that whenever (ii) has been verified for a class of C$^*$-algebras it has also been possible to obtain unitary implementation. However, there is no existing proof of a general implication from (ii) to (iii).

**Reserved for Publication Footnotes**

Various special classes of separable nuclear C$^*$-algebras have been examined. Using Elliott's $K$-theoretic classification result, [8], Phillips and Raeburn showed that two sufficiently close separable approximately finite dimensional (AF) C$^*$-algebras are necessarily isomorphic. Furthermore, using ideas which date back to Bratteli, [9], and Powers, [10], they showed that this isomorphism must be spatially implemented, [11]. In [4], E.C. obtained a near inclusion result for finite dimensional C$^*$-algebras: when $\gamma$ is sufficiently small, a near inclusion $F \subseteq_\gamma B$ of a finite dimensional C$^*$-algebra $F$ can be spatially perturbed to a proper inclusion $uFu^* \subseteq B$ by a unitary $u \in C^*(B, F, I_H)$ suitably close to the identity. There is also an estimate on $\|u - I_H\|$ in terms of $\gamma$ which is independent of the structure of $F$. With this he went on to show that a C$^*$-algebra sufficiently close to a separable AF C$^*$-algebra is again separable and AF. Other classes of C$^*$-algebras for which a positive answer to the Kadison–Kastler problem has been obtained include the separable continuous trace C$^*$-algebras, [12], and certain extensions of continuous trace or AF C$^*$-algebras by compact operators, [13].

Our main work on the Kadison–Kastler problem is summarized in Theorems 3 and 4. The methods developed to prove these two results can also be applied to near inclusions of C$^*$-algebras. The last section of the paper is devoted to this topic and its uses in the currently evolving classification program for C$^*$-algebras.

## Definitions and basic properties

Given two C$^*$-algebras $A$ and $B$, there are in general many different norms that can be placed on the algebraic tensor product $A \otimes B$ so that the norm completions are again C$^*$-algebras. This situation was originally investigated by Takesaki, [14], who obtained examples of distinct C$^*$-norms on $C^*(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2)$. In the same paper, he was able to show that there is a minimal C$^*$-norm on any algebraic tensor product $A \otimes B$, and we denote the resulting completion by $A \otimes_{\min} B$. The defining property of a nuclear C$^*$-algebra $A$ is that there is a unique C$^*$-norm on $A \otimes B$ for each C$^*$-algebra $B$. Amongst all C$^*$-algebras the nuclear ones have an exceptionally rich structure, and consequently they have received considerable attention since their introduction in [14]. Subsequently, many characterizations of nuclear C$^*$-algebras have been obtained, two of which are of particular relevance to our work. The first involves complete boundedness for which we now review the definition.

We denote the algebra of $n \times n$ scalar matrices by $\mathbb{M}_n$. Each bounded map $\phi\colon\ A \to \mathbb{B}(H)$ has a bounded amplification to $\phi_n\colon\ A \otimes \mathbb{M}_n \to \mathbb{B}(H) \otimes \mathbb{M}_n$, $n \geq 1$, given by $\phi_n((a_{ij})) = (\phi(a_{ij}))$, $(a_{ij}) \in A \otimes \mathbb{M}_n$. Then $\phi$ is completely positive if each $\phi_n$ is positive, $n \geq 1$, and completely bounded if $\sup\{\|\phi_n\|\colon\ n \geq 1\} < \infty$, whereupon this quantity defines the completely bounded norm $\|\phi\|_{\text{cb}}$, [15]. Our first required characterization of nuclearity is due to Choi and Effros, [16].

**Theorem 1.** *A C*-algebra $A$ is nuclear if and only if there exist matrix algebras $\mathbb{M}_{n_\lambda}$ and nets of completely positive contractions*

$$A \xrightarrow{\phi_\lambda} \mathbb{M}_{n_\lambda} \xrightarrow{\psi_\lambda} A$$

*so that*

$$\lim_\lambda \|a - \psi_\lambda(\phi_\lambda(a))\| = 0, \qquad a \in A.$$

The second characterization requires the concept of an approximate diagonal, [17]. This is a bounded net $\{x_\alpha\}$ in the projective tensor product $A \widehat{\otimes} A$ satisfying the two conditions

(i) $\lim_\alpha \|a x_\alpha - x_\alpha a\|_{A \widehat{\otimes} A} = 0, \quad a \in A,$

(ii) $\lim_\alpha \|m(x_\alpha)a - a\| = 0, \quad a \in A,$

where $m\colon\ A \widehat{\otimes} A \to A$ is the bounded multiplication map $a_1 \otimes a_2 \mapsto a_1 a_2$. We recall that the projective norm on an algebraic tensor product $A \otimes B$ of Banach algebras is defined as follows. For each $x \in A \otimes B$,

$$\|x\|_{A \widehat{\otimes} B} = \inf \sum_{i=1}^n \|a_i\| \|b_i\|$$

taken over all possible representations $x = \sum_{i=1}^n a_i \otimes b_i$ as a finite sum of elementary tensors. The projective tensor product $A \widehat{\otimes} B$ is then the completion in this norm. With some trivial exceptions, $A \widehat{\otimes} B$ is not a C$^*$-algebra.

Johnson, [18], had defined amenability for a Banach algebra $A$ by the requirement that all derivations $\delta\colon A \to V$ should be inner for all dual $A$-bimodules $V$, a property enjoyed by the algebra $\ell^1(G)$ (with convolution as multiplication) precisely when the discrete group $G$ is amenable. He then showed this condition to be equivalent to the existence of a bounded approximate diagonal. For C$^*$-algebras, Connes, [19], proved that amenability implies nuclearity, while the reverse implication is due to Haagerup, [20]. Collectively, the work of these authors gives our next characterization of nuclearity, with the special form of the approximate diagonal to be found in [20].

**Theorem 2.** *A C*-algebra $A$ is nuclear if and only if $A$ has a bounded approximate diagonal. Moreover, the approximate diagonal can be chosen from*

$$\operatorname{conv}\{a \otimes a^*\colon\ a \in A,\ \|a\| \leq 1\}.$$

For our results, the existence of the approximate diagonal and its location in this convex set are both of crucial importance. After we have stated our main results, we will explain how these two characterizations are used.

## Main results and methods

Our principal result gives a comprehensive positive answer to the Kadison–Kastler problem for separable nuclear C$^*$-algebras. We split this into two parts, and it is clear from their statements that the first is implied by the second. However, in our approach, Theorem 3 is essential for the proof of Theorem 4.

**Theorem 3.** *Let $A$ and $B$ be C*-algebras on a Hilbert space and suppose that $A$ is separable and nuclear. Let $\gamma$ be a constant satisfying $0 < d(A,B) < \gamma < 2 \times 10^{-6}$, and let $X \subseteq A$ and $Y \subseteq B$ be given finite subsets of the respective unit balls.*

*Then $A$ and $B$ are isomorphic and there exists an isomorphism $\theta\colon\ A \to B$ satisfying*

$$\|\theta(x) - x\|, \|\theta^{-1}(y) - y\| \leq 28\gamma^{1/2}, \qquad x \in X,\ y \in Y.$$

The bound on $d(A,B)$ in this theorem is sufficient to deduce separability and nuclearity of $B$ from these properties for $A$, [4]. The final inequalities in Theorem 3 give us a crucial measure of control over the isomorphism and its inverse, needed for Theorem 4. These cannot be extended from finite subsets to apply to the unit balls due to the examples of [6]. Note that we do not assume units for $A$ or $B$. When the algebras are unital, we do not assume that their units coincide with the identity operator on the underlying Hilbert space, or even that they coincide with each other. However, if an assumption of shared units is made, then improved estimates in the results below become available.

**Theorem 4.** *Let $A$ and $B$ be C*-algebras on a separable Hilbert space and suppose that $A$ is separable and nuclear. If $d(A,B) < 10^{-11}$, then there exists a unitary operator $u \in \mathbb{B}(H)$ so that $B = uAu^*$. This unitary may be chosen from the von Neumann algebra generated by $A, B$ and the identity operator $I_H$.*

The counterexample in the nonseparable nuclear case, [7], and the impossibility of obtaining a good estimate on $\|u - I_H\|$, [6], indicate that this result is optimal for nuclear C*-algebras. We now give a brief overview of the methods employed in proving these theorems, beginning with the first.

The isomorphism from $A$ to $B$ is constructed as the point-norm limit of a sequence of completely positive contractive maps $\alpha_n\colon\ A \to B$, $n \geq 1$, very much in the spirit of the intertwining arguments from the classification program. These must be chosen carefully to ensure that the limit exists and is a $*$-isomorphism. Thus one requirement for these maps is that they exhibit asymptotically the behavior of $*$-homomorphisms, at least on increasing finite sets whose union is dense in $A$. The first step is to construct completely positive contractions from $A$ to $B$ which almost fix the elements of specified finite subsets of the unit ball of $A$. This is achieved by starting with such a finite set $X$, choosing a nearby finite subset $\widetilde{X}$ in the unit ball of $B$, and using Theorem 1 in conjunction with the nuclearity of $B$ to choose a matrix algebra $\mathbb{M}_n$ and completely positive contractions $\phi\colon\ B \to \mathbb{M}_n$, $\psi\colon\ \mathbb{M}_n \to B$ whose composition is close to the identity on $\widetilde{X}$. Arveson's Hahn–Banach Theorem, [21], allows us to extend $\phi$ to $\mathbb{B}(H)$ and restrict back to a completely positive contraction $\tilde{\phi}\colon\ A \to \mathbb{M}_n$, whereupon $\psi \circ \tilde{\phi}\colon\ A \to B$ is close to the identity map on $X \subseteq A$. This procedure only requires a one-sided containment $A \subset_\gamma B$, and leads to the following result.

**Proposition 5.** *Let $A$ and $B$ be C*-algebras with $B$ nuclear, and let $\gamma > 0$ be a constant such that $A \subset_\gamma B$. Given a finite subset $X$ of the unit ball of $A$, there exists a completely positive contraction $\phi\colon\ A \to B$ with $\|\phi(x) - x\| \leq 2\gamma$ for $x \in X$.*

We now return to consideration of close pairs of C*-algebras.

For a given finite subset $Y$ of the unit ball of $A$, we can apply Proposition 5 to the set $X = Y \cup Y^* \cup \{yy^*\colon\ y \in Y \cup Y^*\}$ to find a completely positive contraction $\phi\colon\ A \to B$ satisfying

$$\|\phi(y) - y\| \leq 2\gamma, \quad \|\phi(y)\phi(y^*) - \phi(yy^*)\| \leq 6\gamma, \qquad y \in Y \cup Y^*,$$

so that $\phi$ acts approximately as a $*$-homomorphism on $Y$ up to a small tolerance of $6\gamma$. By working on an increasing sequence of finite sets with dense union in the unit ball of $A$, we construct a sequence of maps $\phi_n\colon\ A \to B$, $n \geq 1$, acting multiplicatively on the related finite subsets up to a tolerance of $6\gamma$. The second step is to modify these maps so that these tolerances tend to 0, while ensuring that the limit is a surjective $*$-isomorphism. This is achieved by the following two technical lemmas. Qualitatively they can be described as saying that certain desirable properties on a given finite set $X$ can be obtained by knowing weaker properties on a larger finite set $Y$. In both cases the set $Y$ arises from the components of suitably chosen elements of the approximate diagonal of Theorem 2. The following definitions will be helpful in stating them. Given C*-algebras $A$ and $D$, a completely positive contraction $\phi\colon\ A \to D$, a constant $\varepsilon > 0$, and a finite subset $X$ of the unit ball of $A$, we say that $\phi$ is an $(X, \varepsilon)$-approximate $*$-homomorphism if

$$\|\phi(x)\phi(x^*) - \phi(xx^*)\| \leq \varepsilon, \qquad x \in X \cup X^*,$$

For bounded maps $\phi_1, \phi_2\colon\ A \to D$, $\phi_1 \approx_{X,\varepsilon} \phi_2$ will mean $\|\phi_1(x) - \phi_2(x)\| \leq \varepsilon$ for $x \in X$, a quantitative formulation of closeness of two maps on a set $X$.

**Lemma 6.** *Let $A$ be a nuclear C*-algebra. Given a finite subset $X$ of the unit ball of $A$ and positive constants $\varepsilon, \gamma$ with $\gamma \leq 1/17$, there exists a finite subset $Y$ of the unit ball of $A$ with the following property. For each C*-algebra $D$ and each $(Y, \gamma)$-approximate $*$-homomorphism $\phi\colon\ A \to D$, there exists an $(X, \varepsilon)$-approximate $*$-homomorphism $\psi\colon\ A \to D$ so that $\|\phi - \psi\| \leq 12\gamma^{1/2}$.*

In the case of the algebra $C^*(G)$ for a discrete amenable group $G$, the finite set $Y$ can be taken to be the image in $C^*(G)$ of a suitable Følner set for $G$. Thus it can be helpful to view the finite sets $Y$ in Lemmas 6 and 7 as generalized forms of such sets.

Proposition 5 allows us to construct $(Y, 6\gamma)$-approximate $*$-homomorphisms which can then be perturbed slightly using Lemma 6 to yield $(X, \varepsilon)$-approximate $*$-homomorphisms. In order to handle the problem of point-norm convergence in Theorem 3, the approximate $*$-homomorphisms must be perturbed again using conjugation by unitaries. The second lemma is the technical device needed to achieve this.

**Lemma 7.** *Let $A$ be a nuclear C*-algebra. Given a finite subset $X$ of the unit ball of $A$ and positive constants $\varepsilon, \gamma$ with $\gamma \leq 13/150$, there exist a finite subset $Y$ of the unit ball of $A$ and $\delta > 0$ with the following property. Given a C*-algebra $D$ and two $(Y, \delta)$-approximate $*$-homomorphisms $\phi_1, \phi_2\colon\ A \to D$ such that $\phi_1 \approx_{Y,\gamma} \phi_2$, there exists a unitary $u$ in the unitization of $D$ so that $\|u - 1\| \leq 3\gamma$ and $\phi_1 \approx_{X,\varepsilon} u\phi_2 u^*$.*

We now turn to the unitary implementation of Theorem 4. Our argument for this, inspired by work of Bratteli on AF-algebras, [9], is more technically involved than the proofs of the previous two lemmas. Using results from [4], it can be reduced to the special case where $A$ and $B$ have the same $*$-strong closures $M$. Fixing a sequence of finite sets $X_n$ in the unit ball of $A$ with dense union, Theorem 3 and Lemma 7 allow us to find isomorphisms $\theta_n\colon\ A \to B$ close to the identity on $X_n$, and unitaries $u_n \in M$ which almost implement $\theta_n$ on $X_n$. The $*$-strong convergence of the sequence $\{u_n\}_{n=1}^\infty$ would guarantee that the limit is a unitary $u$ satisfying $A = uBu^*$, but we cannot expect convergence initially. To make it converge in the $*$-strong topology requires further modification by unitaries, and also consideration of finite sets of vectors. In order to ensure that the property of almost implementing $\theta_n$ is preserved, the unitary $v$ which multiplies $u_n$ must be chosen to approximately commute with suitable finite sets of operators. The details are given in a third lemma in the same spirit as the previous two. It is immediately recognizable as a variant of the Kaplansky density theorem. The condition $\|u - 1\| \leq \alpha < 2$ is included to ensure a gap in the spectrum of $u$, allowing us to take a continuous logarithm. This is necessary for our proof.

**Lemma 8.** *Let $A$ be a unital nuclear C*-algebra faithfully and non-degenerately represented on a Hilbert space $H$, and let $M$ denote its $*$-strong closure. Let $X$ be a finite subset of the unit ball of $A$, and let constants $\varepsilon, \mu > 0$ and $0 < \alpha < 2$ be given. Then there exists a finite subset $Y$ of the unit ball of $A$ and $\delta > 0$ with the following property. Given a finite subset $S$ of the unit ball of $H$ and a unitary $u \in M$ satisfying $\|u - 1\| \leq \alpha$ and*

$$\|uy - yu\| < \delta, \qquad y \in Y,$$

*there exists a unitary $v \in A$ with $\|v - 1\| \leq \alpha$ such that*

$$\|vx - xv\| < \varepsilon, \qquad x \in X,$$

*and*

$$\|v\xi - u\xi\| < \mu, \quad \|v^*\xi - u^*\xi\| < \mu, \qquad \xi \in S.$$

Note that this result applies simultaneously to all unitaries satisfying the hypotheses and to all finite subsets $S$ of $H$. This is essential for the proof of Theorem 4 since $Y$ must be constructed before knowing the particular unitary $u \in M$ and the particular subset $S$ of $H$ to which the lemma will be applied.

## Near inclusions

Nuclearity of $B$ in Proposition 5 is only used to obtain the completely positive contractions of its conclusion, but not subsequently in embedding $A$ into $B$. If these maps can be found by other methods in different situations, where $B$ need not be nuclear, then Lemmas 6 and 7 can be used to construct embeddings of $A$ into $B$ provided that $A$ is separable and nuclear. The maps of Proposition 5 are already known to exist for unital algebras $A$ with an approximately inner half flip, [4]. The defining property for such algebras is that to each finite subset $X$ of $A$ and $\varepsilon > 0$ there corresponds a unitary $u \in A \otimes_{\min} A$ such that

$$\|u(I \otimes x)u^* - x \otimes I\|_{A\otimes_{\min} A} < \varepsilon, \quad x \in X.$$

This condition ensures nuclearity of $A$, [22]. We now present another situation where such maps can be constructed.

The nuclear dimension of a C$^*$-algebra, introduced in [23], is a non-commutative version of the covering dimension of a topological space. It is a generalization of the decomposition rank from [24] which includes infinite C$^*$-algebras. Indeed every classifiable Kirchberg algebra has nuclear dimension at most five. Having finite decomposition rank or finite nuclear dimension is a regularity property playing a key role in current work on the classification of C$^*$-algebras by their $K$-theory and large classes of C$^*$-algebras are known to have finite nuclear dimension, [25]. It is defined using the concept of an order zero map, as we now explain. These are maps $\phi\colon F \to A$ between C$^*$-algebras satisfying the following two requirements, the second of which is an orthogonality condition:

(i) $\phi$ is a completely positive contraction;
(ii) if $e, f \in F^+$ with $ef = 0$, then $\phi(e)\phi(f) = 0$.

If $\pi\colon C_0(0,1] \otimes F \to A$ is a $*$-homomorphism, then $\phi(x) = \pi(\mathrm{id}_{(0,1]} \otimes x)$, $x \in F$, is easily verified to be an order zero map. The converse is also true; all order zero maps arise in this way, [26]. The C$^*$-algebras $A$ of nuclear dimension at most $n$ are defined by a variant of the characterization of nuclearity in Theorem 1. For each finite subset $X$ of $A$ and $\varepsilon > 0$, we require an approximate factorization

$$A \xrightarrow{\ \phi\ } F \xrightarrow{\ \psi\ } A$$

by completely positive maps through a finite dimensional C$^*$-algebra $F$ satisfying

(i) $\|\psi(\phi(x)) - x\| < \varepsilon, \qquad x \in X;$
(ii) $\phi$ is a contraction;
(iii) $\psi$ is the sum of $(n+1)$ order zero maps.

For C$^*$-algebras of finite nuclear dimension, the requisite completely positive contractions from $A$ into $B$, as in Proposition 5, are provided by the following result.

**Lemma 9.** *Let $A$ and $B$ be C$^*$-algebras with $A \subset_\gamma B$ for a sufficiently small constant $\gamma > 0$, and let $A$ have nuclear dimension at most $n$. Given any finite subset $X$ of the unit ball of $A$, there exists a completely positive contraction $\phi\colon A \to B$ such that*

$$\|\phi(x) - x\| \le 2(n+1)(2\gamma+\gamma^2)(2+2\gamma+\gamma^2), \qquad x \in X.$$

By using the factorizations appearing in the definition of finite nuclear dimension, Lemma 9 can be established using the following perturbation result for order zero maps. It is also possible to obtain the map $\psi$ below as an order zero map, but at the expense of a larger estimate on $\|\phi - \psi\|_{\mathrm{cb}}$.

**Lemma 10.** *Let $A \subset_\gamma B$ be a near inclusion of C$^*$-algebras on a Hilbert space. Given a finite dimensional C$^*$-algebra $F$ and an order zero map $\phi : F \to A$, there exists a completely positive map $\psi : F \to B$ with*

$$\|\phi - \psi\|_{\mathrm{cb}} \le (2\gamma+\gamma^2)(2+2\gamma+\gamma^2).$$

The results outlined in the foregoing discussion can be summarized as follows.

**Theorem 11.** *Let $A \subset_\gamma B$ be a near inclusion of C$^*$-algebras with $A$ separable. For sufficiently small values of $\gamma$, there exists an embedding of $A$ into $B$ when one of the following three statements holds:*

(i) *$A$ and $B$ are nuclear;*
(ii) *$A$ has an approximately inner half flip;*
(iii) *$A$ has finite nuclear dimension.*

We conclude with one example of how our methods can be applied to strengthen characterizations of inductive limit algebras. An $A\mathbb{T}$-algebra is defined to be a direct limit of algebras of the form $C(\mathbb{T}) \otimes F$, where $F$ is a finite dimensional C$^*$-algebra, while we refer to a C$^*$-algebra of the form $C(\mathbb{T}) \otimes F_1 \oplus C[0,1] \otimes F_2 \oplus F_3$, where $F_1, F_2, F_3$ are finite dimensional C$^*$-algebras, as a basic building block. Elliott, [27], characterized an $A\mathbb{T}$-algebra by the requirement that each finite subset should be arbitrarily close to a basic building block within the algebra. This is an example of a local characterization, studied in considerable generality by Loring, [28]. Our methods allow us to strengthen the characterization by weaking the requirements, and also extend beyond the particular situation discussed here.

**Corollary 12.** *Let $A$ be a separable C$^*$-algebra with the property that for each finite set $X$ in the unit ball, there exists a C$^*$-subalgebra $A_0$ of $A$ which is a basic building block satisfying $d(x, A_0) \le 10^{-6}$ for $x \in X$. Then $A$ is an $A\mathbb{T}$-algebra.*

Similar statements can be obtained for other characterizations of direct limits.

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