MATH 222, FINAL FALL 2001

14 pts per question plus 4 bonus points. Show all work

Q1. If A is an invertible $n \times n$ matrix, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of linearly independent vectors in $\mathbb{R}^n$, then prove that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ is a linearly independent set of vectors.

Q2. Solve the 2×2 matrix equation $AX + B = 3X$, where

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

Q3. If U and V are subspaces of a vector space W, then prove that

$$U \cap V = \{w : \ w \in U \text{ and } w \in V\}$$

is a subspace of W.

Q4. Is

$$\{x^2 - x + 2, \ 2x^2 + 4x - 1, \ x^2 + 5x - 3\}$$

a spanning set for P_3?

Q5. Let B and C be bases for $\mathbb{R}^2$, and let $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ be the transition matrix from B to C. Find C when

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Q6. The linear transformation $L : P_2 \to \mathbb{R}^2$ is defined by

$$L(p(x)) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}.$$

Find the matrix of L when P_2 and $\mathbb{R}^2$ have respectively the bases

$$B = \{x - 1, \ x - 2\}, \quad C = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}.$$

Q7. Find bases for the kernel and range of the linear transformation $L : \mathbb{R}^5 \to \mathbb{R}^3$ defined by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 5 & 7 & 1 & 1 \\ 1 & 3 & 4 & 2 & 1 \end{pmatrix}.$$

Q8. Solve

$$Y' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y, \quad Y(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Q9. In $\mathbb{R}^4$, the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

are orthogonal. Let

$$\mathbf{v} = \begin{pmatrix} 7 \\ 1 \\ -3 \\ -1 \end{pmatrix}.$$

Find the α's in the equation

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4,$$

by taking dot products by the $\mathbf{v}_i$'s.

Q10. In $\mathbb{R}^3$, let

$$S = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Find a basis for $S^\perp$, and then find $s \in S$ and $s^\perp \in S^\perp$ so that

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = s + s^\perp.$$

Q11. Find the point on the line $y = 2x - 1$ which is closest to the point $(1, 4)$.

Q12. Find the cube root of

$$A = \begin{pmatrix} -10 & -18 \\ 9 & 17 \end{pmatrix}.$$

Q13. For 2×2 matrices A and B, prove that AB and BA have the same trace. Is it possible to find A and B so that $I = AB - BA$?

Q14. Prove the Cauchy–Schwarz inequality

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle.$$