MATH 222, TEST 2

Show all steps for credit. Q1–Q4 12 pts, Q5–Q8 13 pts each

Q1. Let \( L : V \to W \) be a linear transformation. Prove that
\[
\dim \ker L + \dim \text{ran} L = \dim V.
\]

Q2. Let \( L : P_2 \to \mathbb{R}^2 \) be a linear transformation such that
\[
L(x + 1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad L(x + 2) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.
\]
Find \( L(x) \), \( L(1) \) and a general formula for \( L(ax + b) \).

Q3. Let \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by
\[
L \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}.
\]
Find the matrix of \( L \) when both the domain and the range have the basis
\[
B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.
\]

Q4. Let
\[
A = \begin{pmatrix} -2 & -6 \\ 3 & 7 \end{pmatrix}.
\]
Find the eigenvalues and eigenvectors.

Q5. For the matrix \( A \) of Q4, solve the system
\[
Y' = AY, \quad Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
You may use any calculations done in Q4 if you wish.
Q6. \( L : P_3 \to P_3 \) is defined by

\[
L(p(x)) = 3p''(x) + (4x + 2)p'(x) - 8p(x).
\]

Find bases for the kernel and range of \( L \).

Q7. A \( 2 \times 2 \) matrix \( A \) has eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \) with respective eigenvectors \( v_1 \) and \( v_2 \). Prove that these eigenvectors form a basis for \( \mathbb{R}^2 \), and then prove that \( A^2 = I_2 \).

Q8. If \( A \) and \( B \) are similar matrices, prove that \( A^n \) and \( B^n \) are similar for each integer \( n \geq 1 \).