MATH 222, TEST 2

Show all steps for credit. Q1–Q4 12 pts, Q5–Q8 13 pts each

Q1. Let $L : V \to W$ be a linear transformation. Prove that

$$\dim \ker L + \dim \operatorname{ran} L = \dim V.$$

Q2. Let $L : P_2 \to \mathbb{R}^2$ be a linear transformation such that

$$L(x+1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad L(x+2) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Find $L(x)$, $L(1)$ and a general formula for $L(ax+b)$.

Q3. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}.$$

Find the matrix of L when both the domain and the range have the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Q4. Let

$$A = \begin{pmatrix} -2 & -6 \\ 3 & 7 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors.

Q5. For the matrix A of Q4, solve the system

$$Y' = AY, \quad Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

You may use any calculations done in Q4 if you wish.

Q6. $L : P_3 \to P_3$ is defined by

$$L(p(x)) = 3p''(x) + (4x + 2)p'(x) - 8p(x).$$

Find bases for the kernel and range of L.

Q7. A 2×2 matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with respective eigenvectors v_1 and v_2. Prove that these eigenvectors form a basis for $\mathbb{R}^2$, and then prove that $A^2 = I_2$.

Q8. If A and B are similar matrices, prove that A^n and B^n are similar for each integer $n \geq 1$.