MATH 447, HOMEWORK 10, DUE APR 24

Q1. If f is bounded and measurable on $[a, b]$, let

$$\|f\|_\infty = \inf\{c \geq 0 : |f(x)| \leq c \ a.e.\}.$$

Prove that this is a norm, and that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Q2. Prove that simple functions, step functions and continuous functions of compact support are all dense in $L^p(\mathbb{R})$ for each $p \in (1, \infty)$.

Q3. If $f \in L^1(\mathbb{R})$, prove that

$$\lim_{t\to 0} \int_{\mathbb{R}} |f(x) - f(x + t)|\, dx = 0.$$

(Try step functions first)

Q4. If $1 < p_1 < p_2 < \infty$, find f and g so that

$$f \in L^{p_1}(\mathbb{R}),\ f \notin L^{p_2}(\mathbb{R}),\ g \in L^{p_2}(\mathbb{R}),\ g \notin L^{p_1}(\mathbb{R}).$$

Q5. Let H be a Hilbert space with an orthonormal basis $\{x_n\}_{n\geq 1}$. Prove that $\sum \alpha_n x_n$ converges if and only if $\sum \alpha_n^2$ converges.