MATH 447, HOMEWORK 2, DUE THURSDAY JAN 31st

Q1. Prove that uniform convergence of $\sum f_n(x)$ on a set X is equivalent to the following statement: given $\varepsilon > 0$, there exists N such that

$$\sup_{x\in X}\left|\sum_{k=m}^{n} f_k(x)\right| < \varepsilon$$

for all $N \le m \le n$.

Q2. Prove uniform convergence of $\sum e^{-nx}$ and the differentiated series on $[r, R]$ for any $0 < r < R$. Sum the first series, and then sum the differentiated series, justifying your steps.

Q3. Review facts about alternating series. Consider $\sum_{n=1}^{\infty}(-1)^{n+1}x^n/n$. Prove that it converges uniformly on $[0,1]$ and justify the statement

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Q4. On a compact interval, let $f_n(x)$, $g_n(x)$ be uniformly convergent sequences of continuous functions. Prove that the sequence $f_n(x)g_n(x)$ converges uniformly.

Q5. Let $f_n(x) = g_n(x) = x + \frac{\sin x}{n}$, $n \ge 1$. Prove that $f_n(x)$ converges uniformly on $\mathbb{R}$ but $f(x)g(x)$ does not. Why does this not contradict Q4?