Q1. Prove that \((X, d)\) is connected if and only if every continuous function \(f : X \to \{0, 1\}\) is constant. Prove that if \(U\) and \(V\) are connected subsets of \(X\) with nonempty intersection then \(U \cup V\) is connected.

Q2. Prove that \((X, d)\) is connected if and only if \(X \times X\) is connected. The metric on the second space is \(d(x_1, x_2) + d(y_1, y_2)\) for points \((x_i, y_i)\).

Q3. Prove that \(\text{span}\{(1 + x)^{-n} : n \geq 1\}\) is uniformly dense in the set of continuous functions \(f \in C[0, \infty)\) satisfying \(\lim_{x \to \infty} f(x) = 0\).

Q4. Let \(T : C[0, 1] \to C[0, 1]\) be defined by
\[
Tf(x) = \int_0^x f(t) \, dt, \quad x \in [0, 1], \quad f \in C[0, 1].
\]
Prove that \(T\) is not a contraction (consider constant functions) but that \(T^2\) is a contraction. Find a fixed point for \(T\).

Q5. Let \(U\) be an open connected set in \(\mathbb{R}^2\). Let \(x_0 \in U\) be fixed, and let \(V\) be the set of points in \(U\) which can be connected to \(x_0\) by a path in \(U\). Prove that \(V\) and \(V^c \cap U\) are both open sets, where \(V^c\) denotes the complement. Deduce that \(U\) is path connected.