MATH 447, HOMEWORK 6, DUE Feb 28th

Q1. Let $f : (0, 1] \to (0, 1]$ be a contraction. Prove that it extends to a contraction $g : [0, 1] \to [0, 1]$.

Q2. Let Σ be a σ–algebra on a set X and let Y be a fixed subset of X. Prove that

$$\{E \cap Y : E \in \Sigma\}$$

is a σ–algebra on Y.

Q3. Let

$$A = \{E \subseteq \mathbb{R} : E \text{ or } E^c \text{ is finite}\}, \ \Sigma = \{E \subseteq \mathbb{R} : E \text{ or } E^c \text{ is countable}\}.$$

Prove that A is an algebra but not a σ–algebra, that Σ is a σ–algebra and is the smallest one containing A.

Now define μ by $\mu(E)$ is the number of points in E. Prove that this is a measure on Σ.

Q4. Prove that a compact metric space has a countable dense set.

Q5. Let μ_i, $i \geq 1$, be measures on a σ–algebra Σ. For each $E \in \Sigma$, define

$$\mu(E) = \sum_{i=1}^{\infty} \mu_i(E).$$

Prove that μ is also a measure.