MATH 447, FINAL SPRING 2006

20 pts per question. Name theorems you use or state them.

Q1. State the monotone and dominated convergence theorems.

If $f : [0,1] \to [0,\infty)$ is measurable, prove that

$$\lim_{n\to\infty} \int_0^1 f(x)^n \, dx$$

is either ∞ or $\mu\{x : f(x) = 1\}$.

Q2. If f is a step function on $\mathbb{R}$, prove that

$$\lim_{t\to 0} \int_{\mathbb{R}} |f(x+t) - f(x)| \, dx = 0.$$

Stating any density result that you use, prove the same result for $f \in L^1(\mathbb{R})$.

Q3. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. On $H \times H$ define

$$[(h_1, h_2), (h_3, h_4)] = \langle h_1, h_3 \rangle + \langle h_2, h_4 \rangle, \quad h_i \in H.$$

Prove that $[\cdot, \cdot]$ is an inner product on $H \times H$ and that $H \times H$ is a Hilbert space.

Q4. Let X be the set of continuous functions on $[0,1]$ satisfying

$$\int_0^{1/2} 2f(2x)\,dx = \int_0^1 f(x)\,dx.$$

Prove that X is a $\|\cdot\|_\infty$–norm closed subspace of $C[0,1]$. Prove by direct calculation that $\{e^{nx}\}_{n\geq 0} \in X$, and then prove that $X = C[0,1]$.

Q5. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Given $\varepsilon > 0$, prove that there exists $\delta > 0$ such that

$$\int_A |f(x)|\,dx < \varepsilon$$

whenever A is a measurable set with $m(A) < \delta$. Stating any density theorem that you use, prove the same result for any given $f \in L^1(\mathbb{R})$.