#10

P195 Q42.

Poles when $\cosh z = 0$. $e^{z} + e^{-z} = 0$,

$$e^{z} = -e^{-z}, \quad e^{2z} = -1 = e^{i(2n+1)\pi}$$

$$z = \frac{2n+1 i\pi}{2} = (n+\tfrac{1}{2})i\pi.$$

The ones inside $|z|=5$ are $\frac{1}{2}i\pi$, $\frac{3}{2}i\pi$, $-\frac{1}{2}i\pi$, $-\frac{3}{2}i\pi$.

$$\text{Res}\left(\tfrac{1}{2}i\pi\right) = \lim_{z\to \frac{1}{2}i\pi} \frac{2(z-\frac{1}{2}i\pi)}{1+e^{-2z}} = \left.\frac{2}{-2e^{-2z}}\right|_{z=\frac{1}{2}i\pi} = 1$$

with the same result for the other 3.

$$\int = 2\pi i \sum \text{Res} = 8\pi i.$$

Q43 $\quad z^3 + 2z^2 + 2z = 0$, $\quad z(z^2+2z+2) = 0$

$z = 0$ or $z = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. These are the poles, all simple.

Zeroes are solutions of $z^2+4=0$, $z = \pm 2i$

$$\text{Res } 0 = \left.\frac{z(z^2+4)}{z(z^2+2z+2)}\right|_{z=0} = \frac{4}{2} = 2$$

$$\text{Res } -1+i = \frac{(z+1-i)(z^2+4)}{z(z+1-i)(z+1+i)} = \frac{(-1+i)^2+4}{(-1+i)(2i)} = \frac{4-2i}{-2-2i}$$

$$= \frac{2-i}{-1-i} = \frac{(2-i)(-1+i)}{2} = \frac{-1+3i}{2}.$$

$$\operatorname{Res}(-1-i) = \left.\frac{(z+1+i)(z^2+4)}{z(z+1+i)(z+1-i)}\right|_{z=-1-i}$$

$$= \frac{(-1-i)^2+4}{(-1-i)(-2i)} = \frac{4+2i}{-2+2i} = \frac{2+i}{-1+i}$$

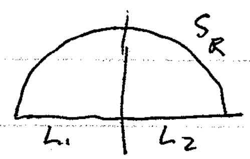
$$= \frac{(2+i)(-1-i)}{2} = \underline{\frac{-1-3i}{2}}.$$

Q45 Use $f^{(2)}\left(\frac{\pi i}{4}\right) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-\pi i/4)^3}\,dz$ with

$f(z) = \sinh 3z$. $f''(z) = 9\sinh 3z = \dfrac{9\left(e^{3z}-e^{-3z}\right)}{2}$

$$\text{so } \int_C \text{---}\,dz = \frac{\pi i\, 9}{2}\left(e^{\frac{3\pi i}{4}} - e^{-\frac{3\pi i}{4}}\right)$$

$$= \frac{2i\,\pi i\, 9}{2}\sin\frac{3\pi}{4} = \underline{-\frac{9\pi}{\sqrt{2}}}$$

Q49 

$$\int_C \frac{\cancel{dz}\,1}{z^4+1}\,dz.$$

$$L_1 = \int_0^R \frac{dx}{x^4+1}, \quad L_2 = \int_{-R}^0 \frac{dx}{x^4+1} = \int_0^R \frac{dt}{t^4+1} \quad (t=-x).$$

$$\text{As } R\to\infty \quad \int_{L_1+L_2} \frac{dz}{z^4+1} \to 2\int_0^\infty \frac{1}{x^4+1}\,dx.$$

$$\left|\int_{S_R} \text{---}\,dz\right| \le \frac{\pi R}{R^4-1} \to 0 \text{ as } R\to\infty.$$

Poles at $e^{i\pi/4}$, $e^{i3\pi/4}$

$$\text{Res}(e^{i\pi/4}) = \lim_{z \to e^{i\pi/4}} \frac{z - e^{i\pi/4}}{z^4+1} = \frac{1}{4e^{i3\pi/4}} \quad \text{(L'Hopital)}$$

and similarly $\text{Res}(e^{i3\pi/4}) = \dfrac{1}{4e^{i9\pi/4}}$

$$2\int_0^\infty \frac{1}{x^4+1}\,dx = \frac{2\pi i}{4}\left[e^{-i3\pi/4} + e^{-i9\pi/4} \right]$$

$$= \frac{2\pi i}{4}\left[\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}} \right] = \pi/\sqrt{2}$$

$$\int_0^\infty \frac{1}{x^4+1}\,dx = \pi/2\sqrt{2}$$

<u>55b</u> We skipped this.

#11

Q51 Put $z = e^{i\theta}$ $\quad d\theta = \dfrac{dz}{iz}$.

$$\int_{|z|=1} \frac{\frac{1}{2i}\left(z^3 - \frac{1}{z^3}\right)}{5 - 3\left(\frac{z+\frac{1}{z}}{2}\right)} \frac{1}{iz}\, dz$$

$$= \int \frac{1}{i^2 z^4} \frac{z^6 - 1}{(10 - 3z - 3/z)}\, dz$$

$$= \int \frac{1 - z^6}{z^3(-3z^2 + 10z - 3)}\, dz$$

$$= \int \frac{z^6 - 1}{z^3(3z^2 - 10z + 3)}\, dz = \int \frac{z^6 - 1}{z^3(3z - 1)(z - 3)}$$

Poles at 0, $\frac{1}{3}$.

$$\operatorname{Res}\left(\tfrac{1}{3}\right) = \left.\frac{z^6 - 1\ (z - \frac{1}{3})}{z^3(3z-1)(z-3)}\right|_{z=\frac{1}{3}} = \frac{\left(\frac{1}{3}\right)^6 - 1}{\left(\frac{1}{3}\right)^3 3\left(-\frac{8}{3}\right)}$$

$$= \frac{1 - 3^6}{-8(3^3)} = \frac{728}{216} = \frac{182}{54} = \frac{91}{27}.$$

Res 0 $\quad \dfrac{z^6}{z^3}$ is analytic so look at $\dfrac{-1}{z^3(3z-1)(z-3)}$

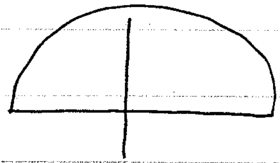
$$= \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} \cdots$$

$$-1 = (3z-1)(z-3)\left(a_{-3} + a_{-2}z + a_{-1}z^2 + \dots\right)$$

$z^0]$ $\quad -1 = 3a_{-3}$, $a_{-3} = -\frac{1}{3}$ $\qquad$ $z]$ $\quad 0 = 3a_{-2} - 10a_{-3}$, $a_{-2} = -\frac{10}{9}$

$z^2]$ $\quad 0 = 3a_{-1} - 10a_{-2} + 3a_{-3}$, $3a_{-1} = \frac{-100}{9} + 1 = -\frac{91}{9}$, $a_{-1} = -\frac{91}{27}$.

Residues cancel so $\int — \, d\theta = 0$.

Q54 

$$\frac{e^{imz}}{(z^2+1)^2}$$

The straight part is

$$\int_0^R \frac{e^{imx}}{(x^2+1)^2}\,dx + \int_{-R}^0 \frac{e^{imx}}{(x^2+1)^2}\,dx$$

$$= \int_0^R \frac{e^{imx}}{(x^2+1)^2}\,dx + \int_0^R \frac{e^{-imt}}{(t^2+1)^2}\,dt \qquad (t=-x)$$

which is $\displaystyle\int_0^R \frac{2\cos mx}{(x^2+1)^2}\,dx$

The curved integral $\to 0$ since $|e^{imz}| = |e^{-my}| \le 1$

and $\left|\dfrac{1}{(z^2+1)^2}\right| \le \dfrac{1}{(R^2-1)^2}$. Just need residue at i.

Double pole.

$$\int_c \frac{e^{imz}}{(z-i)^2(z+i)^2}\,dz = 2\pi i \frac{d}{dz}\left(\frac{e^{imz}}{(z+i)^2}\right)\Bigg|_{z=+i}$$

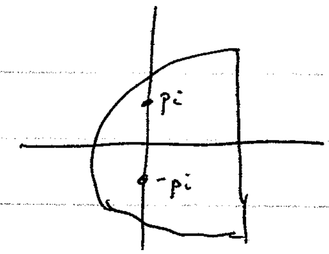
$$= 2\pi i\left(\frac{mi\,e^{imz}(z+i)^2 - e^{imz}\,2(z+i)}{(z+i)^4}\right)\Bigg|_{z=+i}$$

$$= 2\pi i\left(\frac{mi\,e^{-m}(-4) - e^{-m}\,4i}{16}\right)$$

$$= \frac{\pi}{2}\left(me^{-m} + e^{-m}\right)$$

$$\text{so} \int_0^\infty \frac{\cos mx}{(x^2+1)^2}\,dx = \frac{\pi}{4}e^{-m}(m+1)$$

83

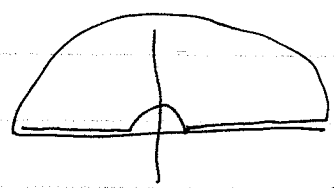


$$\text{Res}\,(ip) = \left.\frac{(z-ip)\,e^{zt}}{(z-ip)(z+ip)}\right|_{z=ip} = \frac{e^{ipt}}{2ip}$$

$$\text{Res}\,(-ip) = \left.\frac{(z+ip)\,e^{zt}}{(z-ip)(z+ip)}\right|_{z=-ip} = \frac{e^{-ipt}}{-2ip}$$

$$2\pi i \sum \text{Res} = 2\pi i \left(\frac{e^{ipt} - e^{-ipt}}{2i\,p} \right) = 2\pi i \, \frac{\sin pt}{p}.$$

84



Use $-\pi/2 < \text{Arg}\, z < \frac{3\pi}{2}$

$$\int_\rho^R \frac{\ln x}{x^2+a^2}\,dx + \int_{-R}^{-\rho} \frac{\ln|x| + i\pi}{x^2+a^2}\,dx$$

$$= 2\int_\rho^R \frac{\ln x}{x^2+a^2}\,dx + i\pi \int_\rho^R \frac{1}{x^2+a^2}\,dx.$$

$$\text{Res}\,(ai) = \left.\frac{(z-ai)\log z}{(z-ai)(z+ai)}\right|_{z=ai} = \frac{\log(ai)}{2ai}$$

$$= \frac{\ln a + i\pi/2}{2ai}$$

$2\pi i \,\text{Res} = \frac{\pi}{a}\left(\ln a + i\pi/2\right)$. Take real parts to get

$$\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi \ln a}{2a}$$

64 Look at $\dfrac{\pi \cos \pi z}{z^4 \sin \pi z}$, poles at $0, \pm n$.

At n $\quad \text{Res} = \lim\limits_{z \to n} \dfrac{\pi \cos \pi z \,(z-n)}{z^4 \sin \pi z}$

$$= \frac{\cos n\pi}{n^4 \cos n\pi} = \frac{1}{n^4}$$

Pole of order 5 at 0.

$$\cancel{\pi}\left(1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} \dots\right) = z^4\left(\pi z - \frac{\pi^3 z^3}{3!} \dots\right)\left(\frac{a_{-5}}{z^5} + \frac{a_{-4}}{z^4} + \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} \dots\right)$$

$$= \cancel{\pi}\left(1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} \dots\right)\left(a_{-5} + a_{-4} z + a_{-3} z^2 + a_{-2} z^3 + a_{-1} z^4 \dots\right)$$

$z^0 \rfloor \quad 1 = a_{-5}$

$z \rfloor \quad 0 = a_{-4}$

$z^2 \rfloor \quad -\dfrac{\pi^2}{2} = -\dfrac{\pi^2}{6} + a_{-3}, \quad a_{-3} = \dfrac{-\pi^2}{3}$

$z^3 \rfloor \quad 0 = a_{-2}$

$z^4 \rfloor \quad \dfrac{\pi^4}{24} = a_{-1} - \dfrac{\pi^2}{6} a_{-3} + \dfrac{\pi^4}{120} a_{-5}$

$$a_{-1} = \pi^4\left(\frac{1}{24} - \frac{1}{18} - \frac{1}{120}\right) = -\frac{\pi^4}{45}$$

$$\therefore\ 2 \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{45}, \quad \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$$