73. Describe graphically the region represented by each of the following:
(a) $1 < |z+i| \leqq 2$, (b) $\text{Re}\{z^2\} > 1$, (c) $|z+3i| > 4$, (d) $|z+2-3i| + |z-2+3i| < 10$.

74. Show that the ellipse $|z+3| + |z-3| = 10$ can be expressed in rectangular form as $x^2/25 + y^2/16 = 1$ [see Problem 13(b)].

AXIOMATIC FOUNDATIONS OF COMPLEX NUMBERS

75. Use the definition of a complex number as an ordered pair of real numbers to prove that if the product of two complex numbers is zero then at least one of the numbers must be zero.

76. Prove the commutative laws with respect to (a) addition, (b) multiplication.

77. Prove the associative laws with respect to (a) addition, (b) multiplication.

78. (a) Find real numbers x and y such that $(c,d)\cdot(x,y) = (a,b)$ where $(c,d) \neq (0,0)$.
(b) How is (x,y) related to the result for division of complex numbers given on Page 2?

79. Prove that

$$(\cos\theta_1, \sin\theta_1)(\cos\theta_2, \sin\theta_2)\cdots(\cos\theta_n, \sin\theta_n) = (\cos[\theta_1+\theta_2+\cdots+\theta_n],\ \sin[\theta_1+\theta_2+\cdots+\theta_n])$$

80. (a) How would you define $(a,b)^{1/n}$ where n is a positive integer?
(b) Determine $(a,b)^{1/2}$ in terms of a and b.

POLAR FORM OF COMPLEX NUMBERS

81. Express each of the following complex numbers in polar form.
(a) $2-2i$, (b) $-1+\sqrt{3}\,i$, (c) $2\sqrt{2}+2\sqrt{2}\,i$, (d) $-i$, (e) -4, (f) $-2\sqrt{3}-2i$, (g) $\sqrt{2}\,i$, (h) $\sqrt{3}/2-3i/2$.
Ans. (a) $2\sqrt{2}$ cis $315°$ or $2\sqrt{2}\,e^{7\pi i/4}$, (b) 2 cis $120°$ or $2e^{2\pi i/3}$, (c) 4 cis $45°$ or $4e^{\pi i/4}$, (d) cis $270°$ or $e^{3\pi i/2}$, (e) 4 cis $180°$ or $4e^{\pi i}$, (f) 4 cis $210°$ or $4e^{7\pi i/6}$, (g) $\sqrt{2}$ cis $90°$ or $\sqrt{2}\,e^{\pi i/2}$, (h) $\sqrt{3}$ cis $300°$ or $\sqrt{3}\,e^{5\pi i/3}$.

82. Show that $2+i = \sqrt{5}\,e^{i\tan^{-1}(1/2)}$.

83. Express in polar form: (a) $-3-4i$, (b) $1-2i$.
Ans. (a) $5\,e^{i(\pi+\tan^{-1}4/3)}$, (b) $\sqrt{5}\,e^{-i\tan^{-1}2}$

84. Graph each of the following and express in rectangular form.
(a) $6(\cos 135° + i\sin 135°)$, (b) 12 cis $90°$, (c) 4 cis $315°$, (d) $2e^{5\pi i/4}$, (e) $5e^{7\pi i/6}$, (f) $3e^{-2\pi i/3}$.
Ans. (a) $-3\sqrt{2}+3\sqrt{2}\,i$, (b) $12i$, (c) $2\sqrt{2}-2\sqrt{2}\,i$, (d) $-\sqrt{2}-\sqrt{2}\,i$, (e) $-5\sqrt{3}/2-(5/2)i$, (f) $-3\sqrt{3}/2-(3/2)i$

85. An airplane travels 150 miles southeast, 100 miles due west, 225 miles 30° north of east, and then 200 miles northeast. Determine (a) analytically and (b) graphically how far and in what direction it is from its starting point. *Ans.* 375 miles, 23° north of east (approx.)

86. Three forces as shown in Fig. 1-41 act in a plane on an object placed at O. Determine (a) graphically and (b) analytically what force is needed to prevent the object from moving. [This force is sometimes called the *equilibrant.*]

87. Prove that on the circle $z = Re^{i\theta}$, $|e^{iz}| = e^{-R\sin\theta}$.

88. (a) Prove that $r_1e^{i\theta_1} + r_2e^{i\theta_2} = r_3e^{i\theta_3}$ where

$$r_3 = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2)}$$

$$\theta_3 = \tan^{-1}\left(\frac{r_1\sin\theta_1 + r_2\sin\theta_2}{r_1\cos\theta_1 + r_2\cos\theta_2}\right)$$

(b) Generalize the result in (a).

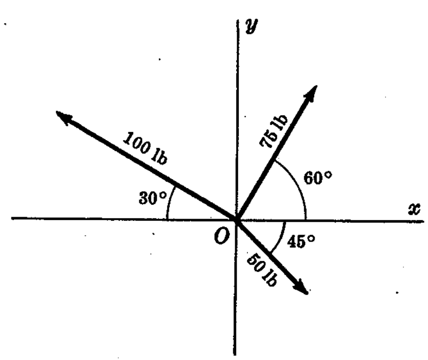

Fig. 1-41

55. Prove that (a) $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$, (b) $\overline{(z_1 z_2 z_3)} = \bar{z}_1 \bar{z}_2 \bar{z}_3$. Generalize these results.

56. Prove that (a) $\overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2$, (b) $|z_1/z_2| = |z_1|/|z_2|$ if $z_2 \neq 0$.

57. Find real numbers x and y such that $2x - 3iy + 4ix - 2y - 5 - 10i = (x + y + 2) - (y - x + 3)i$.
Ans. $x = 1,\ y = -2$

58. Prove that (a) $\text{Re}\{z\} = (z + \bar{z})/2$, (b) $\text{Im}\{z\} = (z - \bar{z})/2i$.

59. Prove that if the product of two complex numbers is zero then at least one of the numbers must be zero.

60. If $w = 3iz - z^2$ and $z = x + iy$, find $|w|^2$ in terms of x and y.
Ans. $x^4 + y^4 + 2x^2y^2 - 6x^2y - 6y^3 + 9x^2 + 9y^2$

GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS. VECTORS.

61. Perform the indicated operations both analytically and graphically.

(a) $(2 + 3i) + (4 - 5i)$ (c) $3(1 + 2i) - 2(2 - 3i)$ (e) $\frac{1}{2}(4 - 3i) + \frac{3}{2}(5 + 2i)$
(b) $(7 + i) - (4 - 2i)$ (d) $3(1 + i) + 2(4 - 3i) - (2 + 5i)$

Ans. (a) $6 - 2i$, (b) $3 + 3i$, (c) $-1 + 12i$, (d) $9 - 8i$, (e) $19/2 + (3/2)i$

62. If z_1, z_2 and z_3 are the vectors indicated in Fig. 1-40, construct graphically:

(a) $2z_1 + z_3$ (c) $z_1 + (z_2 + z_3)$ (e) $\frac{1}{3}z_2 - \frac{3}{4}z_1 + \frac{2}{3}z_3$
(b) $(z_1 + z_2) + z_3$ (d) $3z_1 - 2z_2 + 5z_3$

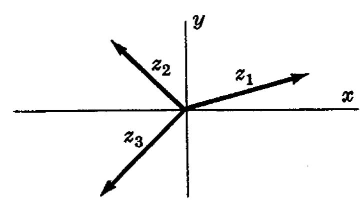

Fig. 1-40

63. If $z_1 = 4 - 3i$ and $z_2 = -1 + 2i$, obtain graphically and analytically (a) $|z_1 + z_2|$, (b) $|z_1 - z_2|$, (c) $\bar{z}_1 - \bar{z}_2$, (d) $|2\bar{z}_1 - 3\bar{z}_2 - 2|$.
Ans. (a) $\sqrt{10}$, (b) $5\sqrt{2}$, (c) $5 + 5i$, (d) 15

64. The position vectors of points A, B and C of triangle ABC are given by $z_1 = 1 + 2i$, $z_2 = 4 - 2i$ and $z_3 = 1 - 6i$ respectively. Prove that ABC is an isosceles triangle and find the lengths of the sides.
Ans. 5, 5, 8

65. Let z_1, z_2, z_3, z_4 be the position vectors of the vertices for quadrilateral $ABCD$. Prove that $ABCD$ is a parallelogram if and only if $z_1 - z_2 - z_3 + z_4 = 0$.

66. If the diagonals of a quadrilateral bisect each other, prove that the quadrilateral is a parallelogram.

67. Prove that the medians of a triangle meet in a point.

68. Let $ABCD$ be a quadrilateral and E, F, G, H the midpoints of the sides. Prove that $EFGH$ is a parallelogram.

69. In parallelogram $ABCD$, point E bisects side AD. Prove that the point where BE meets AC trisects AC.

70. The position vectors of points A and B are $2 + i$ and $3 - 2i$ respectively. (a) Find an equation for line AB. (b) Find an equation for the line perpendicular to AB at its midpoint.

Ans. (a) $z - (2 + i) = t(1 - 3i)$ or $x = 2 + t,\ y = 1 - 3t$ or $3x + y = 7$
(b) $z - (5/2 - i/2) = t(3 + i)$ or $x = 3t + 5/2,\ y = t - 1/2$ or $x - 3y = 4$

71. Describe and graph the locus represented by each of the following: (a) $|z - i| = 2$, (b) $|z + 2i| + |z - 2i| = 6$, (c) $|z - 3| - |z + 3| = 4$, (d) $z(\bar{z} + 2) = 3$, (e) $\text{Im}\{z^2\} = 4$.
Ans. **(a) circle, (b) ellipse, (c) hyperbola, (d) $z = 1$ and $z = -3$, (e) hyperbola**

72. Find an equation for (a) a circle of radius 2 with center at $(-3, 4)$, (b) an ellipse with foci at $(0, 2)$ and $(0, -2)$ whose major axis has length 10.
Ans. (a) $|z + 3 - 4i| = 2$ or $(x + 3)^2 + (y - 4)^2 = 4$, (b) $|z + 2i| + |z - 2i| = 10$

51. If z_1, z_2, z_3 represent vertices of an equilateral triangle, prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

From Fig. 1-39 we see that

$$z_2 - z_1 = e^{\pi i/3}(z_3 - z_1)$$
$$z_1 - z_3 = e^{\pi i/3}(z_2 - z_3)$$

Then by division, $\dfrac{z_2 - z_1}{z_1 - z_3} = \dfrac{z_3 - z_1}{z_2 - z_3}$ or

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

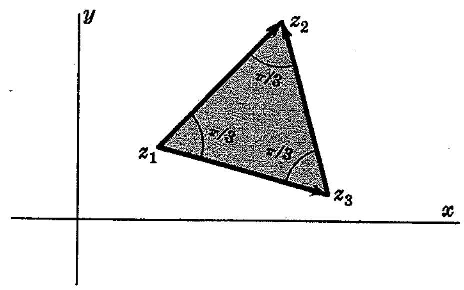

Fig. 1-39

52. Prove that for $m = 2, 3, \ldots$

$$\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\sin\frac{3\pi}{m}\cdots\sin\frac{(m-1)\pi}{m} = \frac{m}{2^{m-1}}$$

The roots of $z^m = 1$ are $z = 1, e^{2\pi i/m}, e^{4\pi i/m}, \ldots, e^{2(m-1)\pi i/m}$. Then we can write

$$z^m - 1 = (z-1)(z - e^{2\pi i/m})(z - e^{4\pi i/m})\cdots(z - e^{2(m-1)\pi i/m})$$

Dividing both sides by $z - 1$ and then letting $z = 1$ [realizing that $(z^m - 1)/(z-1) = 1 + z + z^2 + \cdots + z^{m-1}$] we find

$$m = (1 - e^{2\pi i/m})(1 - e^{4\pi i/m})\cdots(1 - e^{2(m-1)\pi i/m}) \qquad (1)$$

Taking the complex conjugate of both sides of (1) yields

$$m = (1 - e^{-2\pi i/m})(1 - e^{-4\pi i/m})\cdots(1 - e^{-2(m-1)\pi i/m}) \qquad (2)$$

Multiplying (1) by (2) using $(1 - e^{2k\pi i/m})(1 - e^{-2k\pi i/m}) = 2 - 2\cos(2k\pi/m)$, we have

$$m^2 = 2^{m-1}\left(1 - \cos\frac{2\pi}{m}\right)\left(1 - \cos\frac{4\pi}{m}\right)\cdots\left(1 - \cos\frac{2(m-1)\pi}{m}\right) \qquad (3)$$

Since $1 - \cos(2k\pi/m) = 2\sin^2(k\pi/m)$, (3) becomes

$$m^2 = 2^{2m-2}\sin^2\frac{\pi}{m}\sin^2\frac{2\pi}{m}\cdots\sin^2\frac{(m-1)\pi}{m} \qquad (4)$$

Then taking the positive square root of both sides yields the required result.

Supplementary Problems

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

53. Perform each of the indicated operations:

(a) $(4 - 3i) + (2i - 8)$

(b) $3(-1 + 4i) - 2(7 - i)$

(c) $(3 + 2i)(2 - i)$

(d) $(i - 2)\{2(1 + i) - 3(i - 1)\}$

(e) $\dfrac{2 - 3i}{4 - i}$

(f) $(4 + i)(3 + 2i)(1 - i)$

(g) $\dfrac{(2 + i)(3 - 2i)(1 + 2i)}{(1 - i)^2}$

(h) $(2i - 1)^2\left\{\dfrac{4}{1 - i} + \dfrac{2 - i}{1 + i}\right\}$

(i) $\dfrac{i^4 + i^9 + i^{16}}{2 - i^5 + i^{10} - i^{15}}$

(j) $3\left(\dfrac{1 + i}{1 - i}\right)^2 - 2\left(\dfrac{1 - i}{1 + i}\right)^3$

Ans. (a) $-4 - i$ (b) $-17 + 14i$ (c) $8 + i$ (d) $-9 + 7i$ (e) $11/17 - (10/17)i$ (f) $21 + i$ (g) $-15/2 + 5i$ (h) $-11/2 - (23/2)i$ (i) $2 + i$ (j) $-3 - 2i$

54. If $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = \sqrt{3} - 2i$, evaluate each of the following:

(a) $z_1^2 + 2z_1 - 3$

(b) $|2z_2 - 3z_1|^2$

(c) $(z_3 - \bar{z}_3)^5$

(d) $|z_1\bar{z}_2 + z_2\bar{z}_1|$

(e) $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

(f) $\dfrac{1}{2}\left(\dfrac{z_3}{\bar{z}_3} + \dfrac{\bar{z}_3}{z_3}\right)$

(g) $\overline{(z_2 + z_3)(z_1 - z_3)}$

(h) $|z_1^2 + \bar{z}_2^2|^2 + |\bar{z}_3^2 - z_2^2|^2$

(i) $\text{Re}\{2z_1^3 + 3z_2^2 - 5z_3^2\}$

(j) $\text{Im}\{z_1z_2/z_3\}$

Ans. (a) $-1 - 4i$ (b) 170 (c) $1024i$ (d) 12 (e) $3/5$ (f) $-1/7$ (g) $-7 + 3\sqrt{3} + \sqrt{3}\,i$ (h) $765 + 128\sqrt{3}$ (i) -35 (j) $(6\sqrt{3} + 4)/7$