$$\lim_{z \to n} \left\{ \frac{(z-n)}{\sin \pi z} \cdot \frac{\pi \cos \pi z \coth \pi z}{z^3} \right\} = \frac{\coth n\pi}{n^3}$$

Residue at $z = ni$ $(n = \pm 1, \pm 2, \ldots)$ is

$$\lim_{z \to ni} \left\{ \frac{(z-ni)}{\sinh \pi z} \cdot \frac{\pi \cot \pi z \cosh \pi z}{z^3} \right\} = \frac{\coth n\pi}{n^3}$$

Hence by the residue theorem,

$$\oint_{C_N} \frac{\pi \cot \pi z \coth \pi z}{z^3}\, dz = \frac{-7\pi^3}{45} + 4 \sum_{n=1}^{N} \frac{\coth n\pi}{n^3}$$

Taking the limit as $N \to \infty$, we find as in Problem 25 that the integral on the left approaches zero and the required result follows.

Supplementary Problems

RESIDUES AND THE RESIDUE THEOREM

39. For each of the following functions determine the poles and the residues at the poles:

(a) $\dfrac{2z+1}{z^2 - z - 2}$, (b) $\left(\dfrac{z+1}{z-1}\right)^2$, (c) $\dfrac{\sin z}{z^2}$, (d) $\operatorname{sech} z$, (e) $\cot z$.

Ans. (a) $z = -1, 2;\ 1/3, 5/3$
(b) $z = 1;\ 4$
(c) $z = 0;\ 1$
(d) $z = \frac{1}{2}(2k+1)\pi i;\ (-1)^{k+1} i$ where $k = 0, \pm 1, \pm 2, \ldots$
(e) $z = k\pi i;\ 1$ where $k = 0, \pm 1, \pm 2, \ldots$

40. Prove that $\displaystyle\oint_C \frac{\cosh z}{z^3}\, dz = \pi i$ if C is the square with vertices at $\pm 2 \pm 2i$.

41. Show that the residue of $(\csc z \operatorname{csch} z)/z^3$ at $z = 0$ is $-1/60$.

42. Evaluate $\displaystyle\oint_C \frac{e^z\, dz}{\cosh z}$ around the circle C defined by $|z| = 5$. *Ans.* $8\pi i$

43. Find the zeros and poles of $f(z) = \dfrac{z^2+4}{z^3 + 2z^2 + 2z}$ and determine the residues at the poles.
Ans. Zeros: $z = \pm 2i$ Res; at $z = 0$ is 2 Res; at $z = -1+i$ is $-\frac{1}{2}(1-3i)$ Res; at $z = -1-i$ is $-\frac{1}{2}(1+3i)$

44. Evaluate $\displaystyle\oint_C e^{-1/z} \sin(1/z)\, dz$ where C is the circle $|z| = 1$. *Ans.* $2\pi i$

45. Let C be a square bounded by $x = \pm 2$, $y = \pm 2$. Evaluate $\displaystyle\oint_C \frac{\sinh 3z}{(z - \pi i/4)^3}\, dz$. *Ans.* $-9\pi\sqrt{2}/2$

46. Evaluate $\displaystyle\oint_C \frac{2z^2+5}{(z+2)^3 (z^2+4) z^2}\, dz$ where C is (a) $|z - 2i| = 6$, (b) the square with vertices at $1+i,\ 2+i,\ 2+2i,\ 1+2i$.

47. Evaluate $\displaystyle\oint_C \frac{2 + 3 \sin \pi z}{z(z-1)^2}\, dz$ where C is a square having vertices at $3+3i,\ 3-3i,\ -3+3i,\ -3-3i$.
Ans. $-6\pi i$

48. Evaluate $\displaystyle\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z(z^2+1)}\, dz$, $t > 0$ around the square with vertices at $2+2i,\ -2+2i,\ -2-2i,\ 2-2i$. *Ans.* $1 - \cos t$

DEFINITE INTEGRALS

49. Prove that $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

50. Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2}$. *Ans.* $5\pi/288$

51. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin 3\theta}{5 - 3\cos\theta}\,d\theta$. *Ans.* 0

52. Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta}\,d\theta$.

53. Prove that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\,d\theta = \frac{3\pi}{8}$.

54. Prove that if $m > 0$, $\displaystyle\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\,dx = \frac{\pi e^{-m}(1+m)}{4}$.

55. (a) Find the residue of $\dfrac{e^{iz}}{(z^2+1)^5}$ at $z = i$. (b) Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+1)^5}\,dx$.

56. If $a^2 > b^2 + c^2$, prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}$.

57. Prove that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{(5 - 3\cos\theta)^4}\,d\theta = \frac{135\pi}{16{,}384}$.

58. Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^4 + x^2 + 1}$. *Ans.* $\pi\sqrt{3}/6$

59. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2 + 4x + 5)^2}$. *Ans.* $\pi/2$

60. Prove that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

61. Discuss the validity of the following solution to Problem 19. Let $u = (1+i)x/\sqrt{2}$ in the result $\displaystyle\int_0^\infty e^{-u^2}\,du = \tfrac{1}{2}\sqrt{\pi}$ to obtain $\displaystyle\int_0^\infty e^{-ix^2}\,dx = \tfrac{1}{2}(1-i)\sqrt{\pi/2}$ from which $\displaystyle\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \tfrac{1}{2}\sqrt{\pi/2}$ on equating real and imaginary parts.

62. Show that $\displaystyle\int_0^\infty \frac{\cos 2\pi x}{x^4 + x^2 + 1}\,dx = \frac{-\pi}{2\sqrt{3}}\,e^{-\pi/\sqrt{3}}$.

SUMMATION OF SERIES

63. Prove that $\displaystyle\sum_{n=1}^\infty \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\coth\pi + \frac{\pi^2}{4}\operatorname{csch}^2\pi - \frac{1}{2}$.

64. Prove that (a) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$, (b) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^6} = \frac{\pi^6}{945}$.

65. Prove that $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n-1}\, n \sin n\theta}{n^2 + \alpha^2} = \frac{\pi}{2}\,\frac{\sinh \alpha\theta}{\sinh \alpha\pi}$, $-\pi < \theta < \pi$.

66. Prove that $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{12}$.

67. Prove that $\displaystyle\sum_{n=-\infty}^\infty \frac{1}{n^4 + 4a^4} = \frac{\pi}{4a^3}\left\{\frac{\sinh 2\pi a + \sin 2\pi a}{\cosh 2\pi a - \cos 2\pi a}\right\}$.

68. Prove that $\displaystyle\sum_{n=-\infty}^\infty \sum_{m=-\infty}^\infty \frac{1}{(m^2 + a^2)(n^2 + b^2)} = \frac{\pi^2}{ab}\coth\pi a \coth\pi b$.