The rule is valid if a and b are constants, α is a real parameter such that $\alpha_1 \leqq \alpha \leqq \alpha_2$ where α_1 and α_2 are constants, and $F(x,\alpha)$ is continuous and has a continuous partial derivative with respect to α for $a \leqq x \leqq b$, $\alpha_1 \leqq \alpha \leqq \alpha_2$. It can be extended to cases where the limits a and b are infinite or dependent on α.

SUMMATION OF SERIES

The residue theorem can often be used to sum various types of series. The following results are valid under very mild restrictions on $f(z)$ which are generally satisfied whenever the series converge. See Problems 24-32, and 38.

1. $\sum_{-\infty}^{\infty} f(n) \;=\; -\{\text{sum of residues of } \pi \cot \pi z\, f(z) \text{ at all the poles of } f(z)\}$
2. $\sum_{-\infty}^{\infty} (-1)^n f(n) \;=\; -\{\text{sum of residues of } \pi \csc \pi z\, f(z) \text{ at all the poles of } f(z)\}$
3. $\sum_{-\infty}^{\infty} f\left(\frac{2n+1}{2}\right) \;=\; \{\text{sum of residues of } \pi \tan \pi z\, f(z) \text{ at all the poles of } f(z)\}$
4. $\sum_{-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right) \;=\; \{\text{sum of residues of } \pi \sec \pi z\, f(z) \text{ at all the poles of } f(z)\}$

useful.

MITTAG-LEFFLER'S EXPANSION THEOREM

1. Suppose that the only singularities of $f(z)$ in the finite z plane are the simple poles $a_1, a_2, a_3, \ldots$ arranged in order of increasing absolute value.
2. Let the residues of $f(z)$ at $a_1, a_2, a_3, \ldots$ be $b_1, b_2, b_3, \ldots$.
3. Let C_N be circles of radius R_N which do not pass through any poles and on which $|f(z)| < M$, where M is independent of N and $R_N \to \infty$ as $N \to \infty$.

Then *Mittag-Leffler's expansion theorem* states that

$$f(z) \;=\; f(0) + \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{z-a_n} + \frac{1}{a_n} \right\}$$

SOME SPECIAL EXPANSIONS

1. $\csc z \;=\; \frac{1}{z} - 2z\left(\frac{1}{z^2-\pi^2} - \frac{1}{z^2-4\pi^2} + \frac{1}{z^2-9\pi^2} - \cdots\right)$
2. $\sec z \;=\; \pi\left(\frac{1}{(\pi/2)^2-z^2} - \frac{3}{(3\pi/2)^2-z^2} + \frac{5}{(5\pi/2)^2-z^2} - \cdots\right)$
3. $\tan z \;=\; 2z\left(\frac{1}{(\pi/2)^2-z^2} + \frac{1}{(3\pi/2)^2-z^2} + \frac{1}{(5\pi/2)^2-z^2} + \cdots\right)$
4. $\cot z \;=\; \frac{1}{z} + 2z\left(\frac{1}{z^2-\pi^2} + \frac{1}{z^2-4\pi^2} + \frac{1}{z^2-9\pi^2} + \cdots\right)$
5. $\operatorname{csch} z \;=\; \frac{1}{z} - 2z\left(\frac{1}{z^2+\pi^2} - \frac{1}{z^2+4\pi^2} + \frac{1}{z^2+9\pi^2} - \cdots\right)$
6. $\operatorname{sech} z \;=\; \pi\left(\frac{1}{(\pi/2)^2+z^2} - \frac{3}{(3\pi/2)^2+z^2} + \frac{5}{(5\pi/2)^2+z^2} - \cdots\right)$
7. $\tanh z \;=\; 2z\left(\frac{1}{z^2+(\pi/2)^2} + \frac{1}{z^2+(3\pi/2)^2} + \frac{1}{z^2+(5\pi/2)^2} + \cdots\right)$
8. $\coth z \;=\; \frac{1}{z} + 2z\left(\frac{1}{z^2+\pi^2} + \frac{1}{z^2+4\pi^2} + \frac{1}{z^2+9\pi^2} + \cdots\right)$

DEFINITE INTEGRALS

49. Prove that $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

50. Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2}$. *Ans.* $5\pi/288$

51. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin 3\theta}{5 - 3\cos\theta}\, d\theta$. *Ans.* 0

52. Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta}\, d\theta$.

53. Prove that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta = \frac{3\pi}{8}$.

54. Prove that if $m > 0$, $\displaystyle\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\, dx = \frac{\pi e^{-m}(1+m)}{4}$.

55. (*a*) Find the residue of $\dfrac{e^{iz}}{(z^2+1)^5}$ at $z = i$. (*b*) Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+1)^5}\, dx$.

56. If $a^2 > b^2 + c^2$, prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2-c^2}}$.

57. Prove that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{(5 - 3\cos\theta)^4}\, d\theta = \frac{135\pi}{16{,}384}$.

58. Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^4+x^2+1}$. *Ans.* $\pi\sqrt{3}/6$

59. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+4x+5)^2}$. *Ans.* $\pi/2$

60. Prove that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$.

61. Discuss the validity of the following solution to Problem 19. Let $u = (1+i)x/\sqrt{2}$ in the result $\displaystyle\int_0^\infty e^{-u^2}\, du = \tfrac{1}{2}\sqrt{\pi}$ to obtain $\displaystyle\int_0^\infty e^{-ix^2}\, dx = \tfrac{1}{2}(1-i)\sqrt{\pi/2}$ from which $\displaystyle\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \tfrac{1}{2}\sqrt{\pi/2}$ on equating real and imaginary parts.

62. Show that $\displaystyle\int_0^\infty \frac{\cos 2\pi x}{x^4+x^2+1}\, dx = \frac{-\pi}{2\sqrt{3}} e^{-\pi/\sqrt{3}}$.

SUMMATION OF SERIES

63. Prove that $\displaystyle\sum_{n=1}^\infty \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\coth\pi + \frac{\pi^2}{4}\operatorname{csch}^2\pi - \frac{1}{2}$.

64. Prove that (*a*) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$, (*b*) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^6} = \frac{\pi^6}{945}$.

65. Prove that $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n-1}\, n\sin n\theta}{n^2+\alpha^2} = \frac{\pi}{2}\frac{\sinh\alpha\theta}{\sinh\alpha\pi}$, $-\pi < \theta < \pi$.

66. Prove that $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{12}$.

67. Prove that $\displaystyle\sum_{n=-\infty}^\infty \frac{1}{n^4+4a^4} = \frac{\pi}{4a^3}\left\{\frac{\sinh 2\pi a + \sin 2\pi a}{\cosh 2\pi a - \cos 2\pi a}\right\}$.

68. Prove that $\displaystyle\sum_{n=-\infty}^\infty \sum_{m=-\infty}^\infty \frac{1}{(m^2+a^2)(n^2+b^2)} = \frac{\pi^2}{ab}\coth\pi a\coth\pi b$.

MITTAG-LEFFLER'S EXPANSION THEOREM

69. Prove that $\csc z = \frac{1}{z} - 2z\left(\frac{1}{z^2-\pi^2} - \frac{1}{z^2-4\pi^2} + \frac{1}{z^2-9\pi^2} - \cdots\right)$.

70. Prove that $\operatorname{sech} z = \pi\left(\frac{1}{(\pi/2)^2+z^2} - \frac{3}{(3\pi/2)^2+z^2} + \frac{5}{(5\pi/2)^2+z^2} - \cdots\right)$.

71. (a) Prove that $\tan z = 2z\left(\frac{1}{(\pi/2)^2-z^2} + \frac{1}{(3\pi/2)^2-z^2} + \frac{1}{(5\pi/2)^2-z^2} + \cdots\right)$.

(b) Use the result in (a) to show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

72. Prove the expansions (a) 2, (b) 4, (c) 5, (d) 7, (e) 8 on Page 175.

73. Prove that $\sum_{k=1}^{\infty} \frac{1}{z^2+4k^2\pi^2} = \frac{1}{2z}\left\{\frac{1}{2} - \frac{1}{z} + \frac{1}{e^z-1}\right\}$.

74. Prove that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$.

MISCELLANEOUS PROBLEMS

75. Prove that Cauchy's theorem and integral formulas can be obtained as special cases of the residue theorem.

76. Prove that the sum of the residues of the function $\frac{2z^5-4z^2+5}{3z^6-8z+10}$ at all the poles is 2/3.

77. If n is a positive integer, prove that $\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\,d\theta = \frac{2\pi}{n!}$.

78. Evaluate $\oint_C z^3 e^{1/z}\,dz$ around the circle C with equation $|z-1| = 4$. *Ans.* 1/24

79. Prove that under suitably stated conditions on the function:

(a) $\int_0^{2\pi} f(e^{i\theta})\,d\theta = 2\pi f(0)$, (b) $\int_0^{2\pi} f(e^{i\theta})\cos\theta\,d\theta = -\pi f'(0)$.

80. Show that (a) $\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\,d\theta = 2\pi$

(b) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\cos\theta\,d\theta = \pi$.

81. Prove that $\int_0^{\infty} \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{a}{2} - \frac{1}{2a}$.

[*Hint.* Integrate $e^{aiz}/(e^{2\pi z}-1)$ around a rectangle with vertices at $0, R, R+i, i$ and let $R \to \infty$.]

82. Prove that $\int_0^{\infty} \frac{\sin ax}{e^x+1}\,dx = \frac{1}{2a} - \frac{\pi}{2\sinh \pi a}$.

83. If a, p and t are positive constants, prove that $\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{z^2+p^2}\,dz = \frac{\sin pt}{p}$.

84. Prove that $\int_0^{\infty} \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi \ln a}{2a}$.

85. If $-\pi < a < \pi$, prove that $\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\sinh ax}{\sinh \pi x}\,dx = \frac{\sin a}{\cos a + \cosh \lambda}$.

86. Prove that $\displaystyle\int_0^\infty \frac{dx}{(4x^2+\pi^2)\cosh x} = \frac{\ln 2}{2\pi}$.

87. Prove that (a) $\displaystyle\int_0^\infty \frac{\ln x}{x^4+1}\,dx = \frac{-\pi^2\sqrt{2}}{16}$, (b) $\displaystyle\int_0^\infty \frac{(\ln x)^2}{x^4+1}\,dx = \frac{3\pi^3\sqrt{2}}{64}$.

[*Hint.* Consider $\displaystyle\oint_C \frac{(\ln z)^2}{z^4+1}\,dz$ around a semicircle properly indented at $z=0$.]

88. Evaluate $\displaystyle\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx$. *Ans.* $-\pi/4$

89. Prove that if $|a|<1$ and $b>0$, $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh x}\cos bx\,dx = \frac{\pi}{2}\left(\frac{\sin a\pi}{\cos a\pi + \cosh b\pi}\right)$.

90. Prove that if $-1<p<1$, $\displaystyle\int_0^\infty \frac{\cos px}{\cosh x}\,dx = \frac{\pi}{2\cosh(p\pi/2)}$.

91. Prove that $\displaystyle\int_0^\infty \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi\ln 2}{2}$.

92. If $\alpha>0$ and $-\pi/2<\beta<\pi/2$, prove that

(a) $\displaystyle\int_0^\infty e^{-\alpha x^2\cos\beta}\cos(\alpha x^2\sin\beta)\,dx = \tfrac{1}{2}\sqrt{\pi/\alpha}\,\cos(\beta/2)$.

(b) $\displaystyle\int_0^\infty e^{-\alpha x^2\cos\beta}\sin(\alpha x^2\sin\beta)\,dx = \tfrac{1}{2}\sqrt{\pi/\alpha}\,\sin(\beta/2)$.

93. Prove that $\displaystyle\csc^2 z = \sum_{n=-\infty}^{\infty}\frac{1}{(z-n\pi)^2}$.

94. If α and p are real and such that $0<|p|<1$ and $0<|\alpha|<\pi$, prove that

$$\int_0^\infty \frac{x^{-p}\,dx}{x^2+2x\cos\alpha+1} = \left(\frac{\pi}{\sin p\pi}\right)\left(\frac{\sin p\alpha}{\sin\alpha}\right)$$

95. Prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x^2-x^3}} = \frac{2\pi}{\sqrt{3}}$. [*Hint.* Consider the contour of Fig. 7-18.]

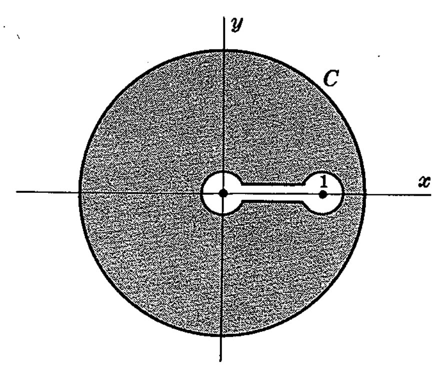

Fig. 7-18

96. Prove the residue theorem for multiply-connected regions.

97. Find sufficient conditions under which the residue theorem (Problem 2) is valid if C encloses infinitely many isolated singularities.

98. Let C be a circle with equation $|z|=4$. Determine the value of the integral

$$\oint_C z^2\csc\frac{1}{z}\,dz$$

if it exists.

99. Give an analytical proof that $\sin\theta \geqq 2\theta/\pi$ for $0\leqq\theta\leqq\pi/2$.
[*Hint.* Consider the derivative of $(\sin\theta)/\theta$, showing that it is a decreasing function.]

100. Prove that $\displaystyle\int_0^\infty \frac{x}{\sinh\pi x}\,dx = \frac{1}{4}$.

101. Verify that the integral around Γ in equation (2) of Problem 22 goes to zero as $R\to\infty$.

102. (a) If r is real, prove that $\displaystyle\int_0^\pi \ln(1-2r\cos\theta+r^2)\,d\theta = \begin{cases} 0 & \text{if } |r|\leqq 1 \\ \pi\ln r^2 & \text{if } |r|\geqq 1\end{cases}$.

(b) Use the result in (a) to evaluate $\displaystyle\int_0^{\pi/2}\ln\sin\theta\,d\theta$ (see Problem 23).