Supplementary Problems

FUNCTIONS AND TRANSFORMATIONS

47. Let $w = f(z) = z(2-z)$. Find the values of w corresponding to (*a*) $z = 1+i$, (*b*) $z = 2-2i$ and graph corresponding values in the w and z planes. *Ans.* (*a*) 2, (*b*) $4+4i$

48. If $w = f(z) = (1+z)/(1-z)$, find (*a*) $f(i)$, (*b*) $f(1-i)$ and represent graphically.
Ans. (*a*) i, (*b*) $-1-2i$

49. If $f(z) = (2z+1)/(3z-2)$, $z \neq 2/3$, find (*a*) $f(1/z)$, (*b*) $f\{f(z)\}$. *Ans.* (*a*) $(2+z)/(3-2z)$, (*b*) z

50. (*a*) If $w = f(z) = (z+2)/(2z-1)$, find $f(0)$, $f(i)$, $f(1+i)$. (*b*) Find the values of z such that $f(z) = i$, $f(z) = 2-3i$. (*c*) Show that z is a single-valued function of w. (*d*) Find the values of z such that $f(z) = z$ and explain geometrically why we would call such values the *fixed* or *invariant points* of the transformation. *Ans.* (*a*) $-2, -i, 1-i$, (*b*) $-i, (2+i)/3$

5(a) **

51. A square S in the z plane has vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Determine the region in the w plane into which S is mapped under the transformations (*a*) $w = z^2$, (*b*) $w = 1/(z+1)$.

52. Discuss Problem 51 if the square has vertices at $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.

53(b) *

53. Separate each of the following into real and imaginary parts, i.e. find $u(x,y)$ and $v(x,y)$ such that $f(z) = u+iv$: (*a*) $f(z) = 2z^2 - 3iz$, (*b*) $f(z) = z + 1/z$, (*c*) $f(z) = (1-z)/(1+z)$, (*d*) $f(z) = z^{1/2}$.

Ans. (*a*) $u = 2x^2 - 2y^2 + 3y$, $v = 4xy - 3x$

(*b*) $u = x + x/(x^2+y^2)$, $v = y - y/(x^2+y^2)$

(*c*) $u = \dfrac{1-x^2-y^2}{(1+x)^2+y^2}$, $v = \dfrac{-2y}{(1+x)^2+y^2}$

(*d*) $u = r^{1/2}\cos\theta/2$, $v = r^{1/2}\sin\theta/2$ where $x = r\cos\theta$, $y = r\sin\theta$

54. If $f(z) = 1/z = u+iv$, construct several members of the families $u(x,y) = \alpha$, $v(x,y) = \beta$ where α and β are constants, showing that they are families of circles.

MULTIPLE-VALUED FUNCTIONS

55. Let $w^3 = z$ and suppose that corresponding to $z=1$ we have $w=1$. (*a*) If we start at $z=1$ in the z plane and make one complete circuit counterclockwise around the origin, find the value of w on returning to $z=1$ for the first time. (*b*) What are the values of w on returning to $z=1$ after $2, 3, 4, \ldots$ complete circuits about the origin? Discuss (*a*) and (*b*) if the paths do not enclose the origin.
Ans. (*a*) $e^{2\pi i/3}$, (*b*) $e^{4\pi i/3}, 1, e^{2\pi i/3}$

56. Let $w = (1-z^2)^{1/2}$ and suppose that corresponding to $z=0$ we have $w=1$. (*a*) If we start at $z=0$ in the z plane and make one complete circuit counterclockwise so as to include $z=1$ but not to include $z=-1$, find the value of w on returning to $z=0$ for the first time. (*b*) What are the values of w if the circuit in (*a*) is repeated over and over again? (*c*) Work parts (*a*) and (*b*) if the circuit includes $z=-1$ but does not include $z=1$. (*d*) Work parts (*a*) and (*b*) if the circuit includes both $z=1$ and $z=-1$. (*e*) Work parts (*a*) and (*b*) if the circuit excludes both $z=1$ and $z=-1$. (*f*) Explain why $z=1$ and $z=-1$ are branch points. (*g*) What lines can be taken as branch lines?

57. Find branch points and construct branch lines for the functions (*a*) $f(z) = \{z/(1-z)\}^{1/2}$, (*b*) $f(z) = (z^2-4)^{1/3}$, (*c*) $f(z) = \ln(z-z^2)$.

THE ELEMENTARY FUNCTIONS

58. Prove that (*a*) $e^{z_1}/e^{z_2} = e^{z_1-z_2}$, (*b*) $|e^{iz}| = e^{-y}$.

59. Prove that there cannot be any finite values of z such that $e^z = 0$.

60. Prove that 2π is a period of e^{iz}. Are there any other periods?

61. Find all values of z for which (*a*) $e^{3z} = 1$, (*b*) $e^{4z} = i$.
Ans. (*a*) $2k\pi i/3$, (*b*) $\frac{1}{8}\pi i + \frac{1}{2}k\pi i$, where $k = 0, \pm1, \pm2, \ldots$.

62. Prove (*a*) $\sin 2z = 2\sin z\cos z$, (*b*) $\cos 2z = \cos^2 z - \sin^2 z$, (*c*) $\sin^2(z/2) = \frac{1}{2}(1-\cos z)$, (*d*) $\cos^2(z/2) = \frac{1}{2}(1+\cos z)$.

63. Prove (*a*) $1 + \tan^2 z = \sec^2 z$, (*b*) $1 + \cot^2 z = \csc^2 z$.

64. If $\cos z = 2$, find (*a*) $\cos 2z$, (*b*) $\cos 3z$. *Ans.* (*a*) 7, (*b*) 26

65. Prove that all the roots of (*a*) $\sin z = a$, (*b*) $\cos z = a$, where $-1 \leqq a \leqq 1$, are real.

LIMITS

89. (a) If $f(z) = z^2 + 2z$, prove that $\lim_{z \to i} f(z) = 2i - 1$.

(b) If $f(z) = \begin{cases} z^2 + 2z & z \neq i \\ 3 + 2i & z = i \end{cases}$, find $\lim_{z \to i} f(z)$ and justify your answer.

90. Prove that $\displaystyle\lim_{z \to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2} = 1 - \tfrac{1}{2}i$.

91. Guess at a possible value for (a) $\displaystyle\lim_{z \to 2+i} \frac{1-z}{1+z}$, (b) $\displaystyle\lim_{z \to 2+i} \frac{z^2 - 2iz}{z^2 + 4}$ and investigate the correctness of your guess.

92. If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, prove that (a) $\lim_{z \to z_0} \{2f(z) - 3i\,g(z)\} = 2A - 3iB$, (b) $\lim_{z \to z_0} \{p\,f(z) + q\,g(z)\} = pA + qB$ where p and q are any constants.

93. If $\lim_{z \to z_0} f(z) = A$, prove that (a) $\lim_{z \to z_0} \{f(z)\}^2 = A^2$, (b) $\lim_{z \to z_0} \{f(z)\}^3 = A^3$. Can you make a similar statement for $\lim_{z \to z_0} \{f(z)\}^n$? What restrictions, if any, must be imposed?

94. Evaluate using theorems on limits. In each case state precisely which theorems are used.

(a) $\displaystyle\lim_{z \to 2i} (iz^4 + 3z^2 - 10i)$

(b) $\displaystyle\lim_{z \to e^{\pi i/4}} \frac{z^2}{z^4 + z + 1}$

(c) $\displaystyle\lim_{z \to i/2} \frac{(2z-3)(4z+i)}{(iz-1)^2}$

(d) $\displaystyle\lim_{z \to i} \frac{z^2+1}{z^6+1}$

(e) $\displaystyle\lim_{z \to 1+i} \left\{\frac{z-1-i}{z^2-2z+2}\right\}^2$

Ans. (a) $-12 + 6i$, (b) $\sqrt{2}\,(1+i)/2$, (c) $-4/3 - 4i$, (d) $1/3$, (e) $-1/4$

95. Find $\displaystyle\lim_{z \to e^{\pi i/3}} (z - e^{\pi i/3}) \left(\frac{z}{z^3+1}\right)$ *Ans.* $1/6 - i\sqrt{3}/6$

96. Prove that if $f(z) = 3z^2 + 2z$, then $\displaystyle\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = 6z_0 + 2$.

97. If $f(z) = \dfrac{2z-1}{3z+2}$, prove that $\displaystyle\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{7}{(3z_0+2)^2}$ provided $z_0 \neq -2/3$.

98. If we restrict ourselves to that branch of $f(z) = \sqrt{z^2+3}$ for which $f(0) = \sqrt{3}$, prove that

$$\lim_{z \to 1} \frac{\sqrt{z^2+3} - 2}{z - 1} = \frac{1}{2}$$

99. Explain exactly what is meant by the statements (a) $\lim_{z \to i} 1/(z-i)^2 = \infty$, (b) $\displaystyle\lim_{z \to \infty} \frac{2z^4+1}{z^4+1} = 2$.

100. Show that (a) $\lim_{z \to \pi/2} (\sin z)/z = 2/\pi$, (b) $\lim_{z \to \pi i/2} z^2 \cosh 4z/3 = \pi^2/8$.

101. Show that if we restrict ourselves to that branch of $f(z) = \tanh^{-1} z$ such that $f(0) = 0$, then $\lim_{z \to -i} f(z) = 3\pi i/4$.

CONTINUITY

102. Let $f(z) = \dfrac{z^2+4}{z-2i}$ if $z \neq 2i$, while $f(2i) = 3 + 4i$. (a) Prove that $\lim_{z \to i} f(z)$ exists and determine its value. (b) Is $f(z)$ continuous at $z = 2i$? Explain. (c) Is $f(z)$ continuous at points $z \neq 2i$? Explain.

103. Answer Problem 102 if $f(2i)$ is redefined as equal to $4i$ and explain why any differences should occur.

104. Prove that $f(z) = z/(z^4+1)$ is continuous at all points inside and on the unit circle $|z| = 1$ except at four points, and determine these points. *Ans.* $e^{(2k+1)\pi i/4}$, $k = 0, 1, 2, 3$

105. If $f(z)$ and $g(z)$ are continuous at $z = z_0$, prove that $3f(z) - 4i\,g(z)$ is also continuous at $z = z_0$.

106. If $f(z)$ is continuous at $z = z_0$, prove that (a) $\{f(z)\}^2$ and (b) $\{f(z)\}^3$ are also continuous at $z = z_0$. Can you extend the result to $\{f(z)\}^n$ where n is any positive integer?

107. Find all points of discontinuity for the following functions.

(a) $f(z) = \dfrac{2z-3}{z^2+2z+2}$, (b) $f(z) = \dfrac{3z^2+4}{z^4-16}$, (c) $f(z) = \cot z$, (d) $f(z) = \dfrac{1}{z} - \sec z$, (e) $f(z) = \dfrac{\tanh z}{z^2+1}$.

Ans. (a) $-1 \pm i$ (c) $k\pi,\ k = 0, \pm1, \pm2, \ldots$
(b) $\pm2, \pm2i$ (d) $0,\ (k+\frac{1}{2})\pi,\ k = 0, \pm1, \pm2, \ldots$ (e) $\pm i,\ (k+\frac{1}{2})\pi i,\ k = 0, \pm1, \pm2, \ldots$

108. Prove that $f(z) = z^2 - 2z + 3$ is continuous everywhere in the finite plane.

109. Prove that $f(z) = \dfrac{z^2+1}{z^3+9}$ is (a) continuous and (b) bounded in the region $|z| \leqq 2$.

110. Prove that if $f(z)$ is continuous in a closed region, it is bounded in the region.

111. Prove that $f(z) = 1/z$ is continuous for all z such that $|z| > 0$, but that it is not bounded.

112. Prove that a polynomial is continuous everywhere in the finite plane.

113. Show that $f(z) = \dfrac{z^2+1}{z^2-3z+2}$ is continuous for all z outside $|z| = 2$.

UNIFORM CONTINUITY

114. Prove that $f(z) = 3z - 2$ is uniformly continuous in the region $|z| \leqq 10$.

115. Prove that $f(z) = 1/z^2$ (a) is not uniformly continuous in the region $|z| \leqq 1$ but (b) is uniformly continuous in the region $\frac{1}{2} \leqq |z| \leqq 1$.

116. Prove that if $f(z)$ is continuous in a closed region $\mathcal{R}$ it is uniformly continuous in $\mathcal{R}$.

SEQUENCES AND SERIES

117. Prove that (a) $\displaystyle\lim_{n\to\infty} \frac{n^2 i^n}{n^3+1} = 0$, (b) $\displaystyle\lim_{n\to\infty} \left(\frac{n}{n+3i} - \frac{in}{n+1}\right) = 1 - i$.

118. Prove that for any complex number z, $\displaystyle\lim_{n\to\infty} (1 + 3z/n^2) = 1$.

119. Prove that $\displaystyle\lim_{n\to\infty} n\left(\frac{1+i}{2}\right)^n = 0$.

120. Prove that $\displaystyle\lim_{n\to\infty} n i^n$ does not exist.

121. If $\displaystyle\lim_{n\to\infty} |u_n| = 0$, prove that $\displaystyle\lim_{n\to\infty} u_n = 0$. Is the converse true? Justify your conclusion.

122. If $\displaystyle\lim_{n\to\infty} a_n = A$ and $\displaystyle\lim_{n\to\infty} b_n = B$, prove that (a) $\displaystyle\lim_{n\to\infty} (a_n + b_n) = A + B$, (b) $\displaystyle\lim_{n\to\infty} (a_n - b_n) = A - B$, (c) $\displaystyle\lim_{n\to\infty} a_n b_n = AB$, (d) $\displaystyle\lim_{n\to\infty} a_n/b_n = A/B$ if $B \neq 0$.

123. Use theorems on limits to evaluate each of the following.

(a) $\displaystyle\lim_{n\to\infty} \frac{in^2 - in + 1 - 3i}{(2n+4i-3)(n-i)}$ (c) $\displaystyle\lim_{n\to\infty} \sqrt{n+2i} - \sqrt{n+i}$

(b) $\displaystyle\lim_{n\to\infty} \left|\frac{(n^2+3i)(n-i)}{in^3 - 3n + 4 - i}\right|$ (d) $\displaystyle\lim_{n\to\infty} \sqrt{n}\{\sqrt{n+2i} - \sqrt{n+i}\}$

Ans. (a) $\frac{1}{2}i$, (b) 1, (c) 0, (d) $\frac{1}{2}i$

124. If $\displaystyle\lim_{n\to\infty} u_n = l$, prove that $\displaystyle\lim_{n\to\infty} \frac{u_1 + u_2 + \cdots + u_n}{n} = l$.

125. Prove that the series $1 + i/3 + (i/3)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} (i/3)^{n-1}$ converges and find its sum.
Ans. $(9+3i)/10$

126. Prove that the series $i - 2i + 3i - 4i + \cdots$ diverges.

127. If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges to A, and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges to B, prove that $\displaystyle\sum_{n=1}^{\infty} (a_n + ib_n)$ converges to $A + iB$. Is the converse true?

128. Investigate the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\omega^n}{5^{n/2}}$ where $\omega = \sqrt{3} + i$. *Ans.* conv.