Supplementary Problems

FUNCTIONS AND TRANSFORMATIONS

47. Let $w = f(z) = z(2-z)$. Find the values of w corresponding to (a) $z = 1+i$, (b) $z = 2-2i$ and graph corresponding values in the w and z planes. *Ans.* (a) 2, (b) $4+4i$

48. If $w = f(z) = (1+z)/(1-z)$, find (a) $f(i)$, (b) $f(1-i)$ and represent graphically.
Ans. (a) i, (b) $-1-2i$

49. If $f(z) = (2z+1)/(3z-2)$, $z \neq 2/3$, find (a) $f(1/z)$, (b) $f\{f(z)\}$. *Ans.* (a) $(2+z)/(3-2z)$, (b) z

50. (a) If $w = f(z) = (z+2)/(2z-1)$, find $f(0)$, $f(i)$, $f(1+i)$. (b) Find the values of z such that $f(z) = i$, $f(z) = 2-3i$. (c) Show that z is a single-valued function of w. (d) Find the values of z such that $f(z) = z$ and explain geometrically why we would call such values the *fixed* or *invariant points* of the transformation. *Ans.* (a) -2, $-i$, $1-i$, (b) $-i$, $(2+i)/3$

51. A square S in the z plane has vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Determine the region in the w plane into which S is mapped under the transformations (a) $w = z^2$, (b) $w = 1/(z+1)$.

52. Discuss Problem 51 if the square has vertices at $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.

53. Separate each of the following into real and imaginary parts, i.e. find $u(x,y)$ and $v(x,y)$ such that $f(z) = u+iv$: (a) $f(z) = 2z^2 - 3iz$, (b) $f(z) = z + 1/z$, (c) $f(z) = (1-z)/(1+z)$, (d) $f(z) = z^{1/2}$.

Ans. (a) $u = 2x^2 - 2y^2 + 3y$, $v = 4xy - 3x$

(b) $u = x + x/(x^2+y^2)$, $v = y - y/(x^2+y^2)$

(c) $u = \dfrac{1-x^2-y^2}{(1+x)^2+y^2}$, $v = \dfrac{-2y}{(1+x)^2+y^2}$

(d) $u = r^{1/2}\cos\theta/2$, $v = r^{1/2}\sin\theta/2$ where $x = r\cos\theta$, $y = r\sin\theta$

54. If $f(z) = 1/z = u+iv$, construct several members of the families $u(x,y) = \alpha$, $v(x,y) = \beta$ where α and β are constants, showing that they are families of circles.

MULTIPLE-VALUED FUNCTIONS

55. Let $w^3 = z$ and suppose that corresponding to $z = 1$ we have $w = 1$. (a) If we start at $z = 1$ in the z plane and make one complete circuit counterclockwise around the origin, find the value of w on returning to $z = 1$ for the first time. (b) What are the values of w on returning to $z = 1$ after $2, 3, 4, \ldots$ complete circuits about the origin? Discuss (a) and (b) if the paths do not enclose the origin.
Ans. (a) $e^{2\pi i/3}$, (b) $e^{4\pi i/3}$, 1, $e^{2\pi i/3}$

56. Let $w = (1-z^2)^{1/2}$ and suppose that corresponding to $z = 0$ we have $w = 1$. (a) If we start at $z = 0$ in the z plane and make one complete circuit counterclockwise so as to include $z = 1$ but not to include $z = -1$, find the value of w on returning to $z = 0$ for the first time. (b) What are the values of w if the circuit in (a) is repeated over and over again? (c) Work parts (a) and (b) if the circuit includes $z = -1$ but does not include $z = 1$. (d) Work parts (a) and (b) if the circuit includes both $z = 1$ and $z = -1$. (e) Work parts (a) and (b) if the circuit excludes both $z = 1$ and $z = -1$. (f) Explain why $z = 1$ and $z = -1$ are branch points. (g) What lines can be taken as branch lines?

57. Find branch points and construct branch lines for the functions (a) $f(z) = \{z/(1-z)\}^{1/2}$, (b) $f(z) = (z^2-4)^{1/3}$, (c) $f(z) = \ln(z-z^2)$.

THE ELEMENTARY FUNCTIONS

58. Prove that (a) $e^{z_1}/e^{z_2} = e^{z_1 - z_2}$, (b) $|e^{iz}| = e^{-y}$.

59. Prove that there cannot be any finite values of z such that $e^z = 0$.

60. Prove that 2π is a period of e^{iz}. Are there any other periods?

61. Find all values of z for which (a) $e^{3z} = 1$, (b) $e^{4z} = i$.
Ans. (a) $2k\pi i/3$, (b) $\frac{1}{8}\pi i + \frac{1}{2}k\pi i$, where $k = 0, \pm 1, \pm 2, \ldots$.

62. Prove (a) $\sin 2z = 2\sin z\cos z$, (b) $\cos 2z = \cos^2 z - \sin^2 z$, (c) $\sin^2(z/2) = \frac{1}{2}(1-\cos z)$, (d) $\cos^2(z/2) = \frac{1}{2}(1+\cos z)$.

63. Prove (a) $1 + \tan^2 z = \sec^2 z$, (b) $1 + \cot^2 z = \csc^2 z$.

64. If $\cos z = 2$, find (a) $\cos 2z$, (b) $\cos 3z$. *Ans.* (a) 7, (b) 26

65. Prove that all the roots of (a) $\sin z = a$, (b) $\cos z = a$, where $-1 \leqq a \leqq 1$, are real.

66. Prove that if $|\sin z| \leqq 1$ for all z, then $|\text{Im}\{z\}| \leqq \ln(\sqrt{2}+1)$.

67. Show that (*a*) $\overline{\sin z} = \sin \bar{z}$, (*b*) $\overline{\cos z} = \cos \bar{z}$, (*c*) $\overline{\tan z} = \tan \bar{z}$.

68. For each of the following functions find $u(x,y)$ and $v(x,y)$ such that $f(z) = u + iv$, i.e. separate into real and imaginary parts: (*a*) $f(z) = e^{3iz}$, (*b*) $f(z) = \cos z$, (*c*) $f(z) = \sin 2z$, (*d*) $f(z) = z^2 e^{2z}$.
Ans. (*a*) $u = e^{-3y}\cos 3x$, $v = e^{-3y}\sin 3x$, (*b*) $u = \cos x \cosh y$, $v = -\sin x \sinh y$, (*c*) $u = \sin 2x \cosh 2y$, $v = \cos 2x \sinh 2y$, (*d*) $u = e^{2x}\{(x^2-y^2)\cos 2y - 2xy \sin 2y\}$, $v = e^{2x}\{2xy\cos 2y + (x^2-y^2)\sin 2y\}$

69. Prove that (*a*) $\sinh(-z) = -\sinh z$, (*b*) $\cosh(-z) = \cosh z$, (*c*) $\tanh(-z) = -\tanh z$.

70. Prove that (*a*) $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$, (*b*) $\cosh 2z = \cosh^2 z + \sinh^2 z$, (*c*) $1 - \tanh^2 z = \text{sech}^2 z$.

71. Prove that (*a*) $\sinh^2(z/2) = \frac{1}{2}(\cosh z - 1)$, (*b*) $\cosh^2(z/2) = \frac{1}{2}(\cosh z + 1)$.

72. Find $u(x,y)$ and $v(x,y)$ such that (*a*) $\sinh 2z = u + iv$, (*b*) $z\cosh z = u + iv$.
Ans. (*a*) $u = \sinh 2x \cos 2y$, $v = \cosh 2x \sin 2y$
(*b*) $u = x\cosh x \cos y - y \sinh x \sin y$, $v = y \cosh x \cos y + x \sinh x \sin y$

73. Find the value of (*a*) $4\sinh(\pi i/3)$, (*b*) $\cosh(2k+1)\pi i/2$, $k = 0, \pm 1, \pm 2, \ldots$, (*c*) $\coth 3\pi i/4$.
Ans. (*a*) $2i\sqrt{3}$, (*b*) 0, (*c*) i

74. (*a*) Show that $\ln\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \left(\frac{4\pi}{3} + 2k\pi\right)i$, $k = 0, \pm 1, \pm 2, \ldots$. (*b*) What is the principal value?
Ans. (*b*) $4\pi i/3$

75. Obtain all the values of (*a*) $\ln(-4)$, (*b*) $\ln(3i)$, (*c*) $\ln(\sqrt{3}-i)$ and find the principal value in each case.
Ans. (*a*) $2\ln 2 + (\pi + 2k\pi)i$, $2\ln 2 + \pi i$. (*b*) $\ln 3 + (\pi/2 + 2k\pi)i$, $\ln 3 + \pi i/2$. (*c*) $\ln 2 + (11\pi/6 + 2k\pi)i$, $\ln 2 + 11\pi i/6$

76. Show that $\ln(z-1) = \frac{1}{2}\ln\{(x-1)^2 + y^2\} + i \tan^{-1} y/(x-1)$, giving restrictions if any.

77. Prove that (*a*) $\cos^{-1} z = \frac{1}{i}\ln(z + \sqrt{z^2-1})$, (*b*) $\cot^{-1} z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$ indicating any restrictions.

78. Prove that (*a*) $\sinh^{-1} z = \ln(z + \sqrt{z^2+1})$, (*b*) $\coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$.

79. Find all the values of (*a*) $\sin^{-1} 2$, (*b*) $\cos^{-1} i$.
Ans. (*a*) $\pm i \ln(2+\sqrt{3}) + \pi/2 + 2k\pi$ (*b*) $-i\ln(\sqrt{2}+1) + \pi/2 + 2k\pi$, $-i\ln(\sqrt{2}-1) + 3\pi/2 + 2k\pi$

80. Find all the values of (*a*) $\cosh^{-1} i$, (*b*) $\sinh^{-1}\{\ln(-1)\}$.
Ans. (*a*) $\ln(\sqrt{2}+1) + \pi i/2 + 2k\pi i$, $\ln(\sqrt{2}-1) + 3\pi i/2 + 2k\pi i$
(*b*) $\ln[(2k+1)\pi + \sqrt{(2k+1)^2\pi^2 - 1}] + \pi i/2 + 2m\pi i$,
$\ln[\sqrt{(2k+1)^2\pi^2 - 1} - (2k+1)\pi] + 3\pi i/2 + 2m\pi i$, $k, m = 0, \pm 1, \pm 2, \ldots$

81. Determine all the values of (*a*) $(1+i)^i$, (*b*) $1^{\sqrt{2}}$.
Ans. (*a*) $e^{-\pi/4 + 2k\pi}\{\cos(\frac{1}{2}\ln 2) + i \sin(\frac{1}{2}\ln 2)\}$, (*b*) $\cos(2\sqrt{2}\,k\pi) + i\sin(2\sqrt{2}\,k\pi)$

82. Find (*a*) $\text{Re}\{(1-i)^{1+i}\}$, (*b*) $|(-i)^{-i}|$.
Ans. (*a*) $e^{\frac{1}{2}\ln 2 - 7\pi/4 - 2k\pi}\cos(7\pi/4 + \frac{1}{2}\ln 2)$, (*b*) $e^{3\pi/2 + 2k\pi}$

83. Find the real and imaginary parts of z^z where $z = x + iy$.

84. Show that (*a*) $f(z) = (z^2-1)^{1/3}$, (*b*) $f(z) = z^{1/2} + z^{1/3}$ are algebraic functions of z.

BRANCH POINTS, BRANCH LINES AND RIEMANN SURFACES

85. Prove that $z = \pm i$ are branch points of $(z^2+1)^{1/3}$.

86. Construct a Riemann surface for the functions (*a*) $z^{1/3}$, (*b*) $z^{1/2}(z-1)^{1/2}$, (*c*) $\left(\frac{z+2}{z-2}\right)^{1/3}$.

87. Show that the Riemann surface for the function $z^{1/2} + z^{1/3}$ has 6 sheets.

88. Construct Riemann surfaces for the functions (*a*) $\ln(z+2)$, (*b*) $\sin^{-1} z$, (*c*) $\tan^{-1} z$.

LIMITS

89. (*a*) If $f(z) = z^2+2z$, prove that $\lim_{z\to i} f(z) = 2i-1$.

(*b*) If $f(z) = \begin{cases} z^2+2z & z \neq i \\ 3+2i & z = i \end{cases}$, find $\lim_{z\to i} f(z)$ and justify your answer.

90. Prove that $\lim_{z\to 1+i} \dfrac{z^2-z+1-i}{z^2-2z+2} = 1 - \tfrac{1}{2}i$.

91. Guess at a possible value for (*a*) $\lim_{z\to 2+i} \dfrac{1-z}{1+z}$, (*b*) $\lim_{z\to 2+i} \dfrac{z^2-2iz}{z^2+4}$ and investigate the correctness of your guess.

92. If $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$, prove that (*a*) $\lim_{z\to z_0} \{2f(z) - 3i\,g(z)\} = 2A - 3iB$, (*b*) $\lim_{z\to z_0} \{p\,f(z) + q\,g(z)\} = pA + qB$ where p and q are any constants.

93. If $\lim_{z\to z_0} f(z) = A$, prove that (*a*) $\lim_{z\to z_0} \{f(z)\}^2 = A^2$, (*b*) $\lim_{z\to z_0} \{f(z)\}^3 = A^3$. Can you make a similar statement for $\lim_{z\to z_0} \{f(z)\}^n$? What restrictions, if any, must be imposed?

94. Evaluate using theorems on limits. In each case state precisely which theorems are used.

(*a*) $\lim_{z\to 2i} (iz^4+3z^2-10i)$

(*b*) $\lim_{z\to e^{\pi i/4}} \dfrac{z^2}{z^4+z+1}$

(*c*) $\lim_{z\to i/2} \dfrac{(2z-3)(4z+i)}{(iz-1)^2}$

(*d*) $\lim_{z\to i} \dfrac{z^2+1}{z^6+1}$

(*e*) $\lim_{z\to 1+i} \left\{\dfrac{z-1-i}{z^2-2z+2}\right\}^2$

Ans. (*a*) $-12+6i$, (*b*) $\sqrt{2}\,(1+i)/2$, (*c*) $-4/3-4i$, (*d*) $1/3$, (*e*) $-1/4$

95. Find $\lim_{z\to e^{\pi i/3}} (z-e^{\pi i/3})\left(\dfrac{z}{z^3+1}\right)$ *Ans.* $1/6 - i\sqrt{3}/6$

96. Prove that if $f(z) = 3z^2+2z$, then $\lim_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0} = 6z_0+2$.

97. If $f(z) = \dfrac{2z-1}{3z+2}$, prove that $\lim_{h\to 0} \dfrac{f(z_0+h)-f(z_0)}{h} = \dfrac{7}{(3z_0+2)^2}$ provided $z_0 \neq -2/3$.

98. If we restrict ourselves to that branch of $f(z) = \sqrt{z^2+3}$ for which $f(0) = \sqrt{3}$, prove that

$$\lim_{z\to 1} \frac{\sqrt{z^2+3}-2}{z-1} = \frac{1}{2}$$

99. Explain exactly what is meant by the statements (*a*) $\lim_{z\to i} 1/(z-i)^2 = \infty$, (*b*) $\lim_{z\to\infty} \dfrac{2z^4+1}{z^4+1} = 2$.

100. Show that (*a*) $\lim_{z\to\pi/2} (\sin z)/z = 2/\pi$, (*b*) $\lim_{z\to\pi i/2} z^2 \cosh 4z/3 = \pi^2/8$.

101. Show that if we restrict ourselves to that branch of $f(z) = \tanh^{-1} z$ such that $f(0) = 0$, then $\lim_{z\to -i} f(z) = 3\pi i/4$.

CONTINUITY

102. Let $f(z) = \dfrac{z^2+4}{z-2i}$ if $z \neq 2i$, while $f(2i) = 3+4i$. (*a*) Prove that $\lim_{z\to i} f(z)$ exists and determine its value. (*b*) Is $f(z)$ continuous at $z=2i$? Explain. (*c*) Is $f(z)$ continuous at points $z\neq 2i$? Explain.

103. Answer Problem 102 if $f(2i)$ is redefined as equal to $4i$ and explain why any differences should occur.

104. Prove that $f(z) = z/(z^4+1)$ is continuous at all points inside and on the unit circle $|z|=1$ except at four points, and determine these points. *Ans.* $e^{(2k+1)\pi i/4}$, $k = 0,1,2,3$

105. If $f(z)$ and $g(z)$ are continuous at $z=z_0$, prove that $3f(z) - 4i\,g(z)$ is also continuous at $z=z_0$.

106. If $f(z)$ is continuous at $z=z_0$, prove that (*a*) $\{f(z)\}^2$ and (*b*) $\{f(z)\}^3$ are also continuous at $z=z_0$. Can you extend the result to $\{f(z)\}^n$ where n is any positive integer?

MISCELLANEOUS PROBLEMS

129. Let $w = \{(4-z)(z^2+4)\}^{1/2}$. If $w=4$ when $z=0$, show that if z describes the curve C of Fig. 2-32, then the value of w at $z=6$ is $-4i\sqrt{5}$.

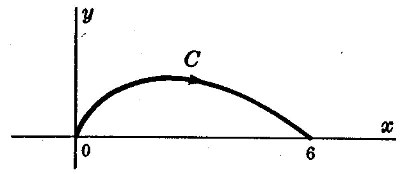

Fig. 2-32

130. Prove that a necessary and sufficient condition for $f(z) = u(x,y) + i\,v(x,y)$ to be continuous at $z = z_0 = x_0 + iy_0$ is that $u(x,y)$ and $v(x,y)$ be continuous at (x_0, y_0).

131. Prove that the equation $\tan z = z$ has only real roots.

132. A student remarked that 1 raised to any power is equal to 1. Was he correct? Explain.

133. Show that $\dfrac{\sin\theta}{2} + \dfrac{\sin 2\theta}{2^2} + \dfrac{\sin 3\theta}{2^3} + \cdots = \dfrac{2\sin\theta}{5-4\cos\theta}$.

134. Show that the relation $|f(x+iy)| = |f(x)+f(iy)|$ is satisfied by $f(z) = \sin z$. Can you find any other functions for which it is true?

135. Prove that $\displaystyle\lim_{z\to\infty} \frac{z^3-3z+2}{z^4+z^2-3z+5} = 0$.

136. Prove that $|\csc z| \leqq 2e/(e^2-1)$ if $|y| \geqq 1$.

137. Show that $\text{Re}\{\sin^{-1} z\} = \frac{1}{2}\{\sqrt{x^2+y^2+2x+1} - \sqrt{x^2+y^2-2x+1}\}$.

138. If $f(z)$ is continuous in a bounded closed region $\mathcal{R}$, prove that (*a*) there exists a positive number M such that for all z in $\mathcal{R}$, $|f(z)| \leqq M$, (*b*) $|f(z)|$ has a *least upper bound* μ in $\mathcal{R}$ and there exists at least one value z_0 in $\mathcal{R}$ such that $|f(z_0)| = \mu$.

139. Show that $|\tanh \pi(1+i)/4| = 1$.

140. Prove that all the values of $(1-i)^{\sqrt{2}\,i}$ lie on a straight line.

141. Evaluate (*a*) $\cosh \pi i/2$, (*b*) $\tanh^{-1}\infty$. *Ans.* (*a*) 0, (*b*) $(2k+1)\pi i/2$, $k = 0, \pm1, \pm2, \ldots$.

142. If $\tan z = u + iv$, show that

$$u = \frac{\sin 2x}{\cos 2x + \cosh 2y}, \qquad v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

143. Evaluate to 3 decimal place accuracy: (*a*) e^{3-2i}, (*b*) $\sin(5-4i)$.

144. Prove $\text{Re}\left\{\dfrac{1+i\tan(\theta/2)}{1-i\tan(\theta/2)}\right\} = \cos\theta$, indicating any exceptional values.

145. If $\displaystyle\lim_{z\to z_0} f(z) = A$ and $\displaystyle\lim_{z\to z_0} g(z) = B \neq 0$, prove that $\displaystyle\lim_{z\to z_0} f(z)/g(z) = A/B$ without first proving that $\displaystyle\lim_{z\to z_0} 1/g(z) = 1/B$.

146. Let $f(z) = \begin{cases} 1 & \text{if } |z| \text{ is rational} \\ 0 & \text{if } |z| \text{ is irrational} \end{cases}$. Prove that $f(z)$ is discontinuous at all values of z.

147. Prove that if $f(z) = u(x,y) + i\,v(x,y)$ is continuous in a region, then (*a*) $\text{Re}\{f(z)\} = u(x,y)$ and (*b*) $\text{Im}\{f(z)\} = v(x,y)$ are continuous in the region.

148. Prove that all the roots of $z\tan z = k$, where $k > 0$, are real.

149. Prove that if the limit of a sequence exists it must be unique.

150. (*a*) Prove that $\displaystyle\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0$.

(*b*) Prove that the series $\displaystyle\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n})$ diverges, thus showing that a series whose nth term approaches zero need not converge.

151. If $z_{n+1} = \frac{1}{2}(z_n + 1/z_n)$, $n = 0, 1, 2, \ldots$ and $-\pi/2 < \arg z_0 < \pi/2$, prove that $\displaystyle\lim_{n\to\infty} z_n = 1$.