Integrating (*1*) with respect to z (treating $\bar{z}$ as constant),

$$\frac{\partial U}{\partial \bar{z}} = \frac{z^3}{24} + \frac{z\bar{z}^2}{8} + F_1(\bar{z}) \tag{2}$$

where $F_1(\bar{z})$ is an arbitrary function of $\bar{z}$. Integrating (*2*) with respect to $\bar{z}$,

$$U = \frac{z^3\bar{z}}{24} + \frac{z\bar{z}^3}{24} + F(\bar{z}) + G(z) \tag{3}$$

where $F(\bar{z})$ is the function obtained by integrating $F_1(\bar{z})$, and $G(z)$ is an arbitrary function of z. Replacing z and $\bar{z}$ by $x+iy$ and $x-iy$ respectively, we obtain

$$U = \tfrac{1}{12}(x^4 - y^4) + F(x-iy) + G(x+iy)$$

Supplementary Problems

DERIVATIVES

43. Using the definition, find the derivative of each function at the indicated points.
(*a*) $f(z) = 3z^2 + 4iz - 5 + i$; $z = 2$. (*b*) $f(z) = \dfrac{2z-i}{z+2i}$; $z = -i$. (*c*) $f(z) = 3z^{-2}$; $z = 1+i$.
Ans. (*a*) $12 + 4i$ (*b*) $-5i$ (*c*) $3/2 + 3i/2$

44. Prove that $\dfrac{d}{dz}(z^2\bar{z})$ does not exist anywhere.

45. Determine whether $|z|^2$ has a derivative anywhere.

46. For each of the following functions determine the singular points, i.e. points at which the function is not analytic. Determine the derivatives at all other points. (*a*) $\dfrac{z}{z+i}$, (*b*) $\dfrac{3z-2}{z^2+2z+5}$.
Ans. (*a*) $-i$, $i/(z+i)^2$; (*b*) $-1 \pm 2i$, $(19 + 4z - 3z^2)/(z^2+2z+5)^2$

CAUCHY-RIEMANN EQUATIONS

47. Verify that the real and imaginary parts of the following functions satisfy the Cauchy-Riemann equations and thus deduce the analyticity of each function:
(*a*) $f(z) = z^2 + 5iz + 3 - i$, (*b*) $f(z) = ze^{-z}$, (*c*) $f(z) = \sin 2z$.

48. Show that the function $x^2 + iy^3$ is not analytic anywhere. Reconcile this with the fact that the Cauchy-Riemann equations are satisfied at $x = 0$, $y = 0$.

49. Prove that if $w = f(z) = u + iv$ is analytic in a region $\mathcal{R}$, then $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial x} = -i\dfrac{\partial w}{\partial y}$.

50. (*a*) Prove that the function $u = 2x(1-y)$ is harmonic. (*b*) Find a function v such that $f(z) = u + iv$ is analytic [i.e. find the conjugate function of u]. (*c*) Express $f(z)$ in terms of z.
Ans. (*b*) $2y + x^2 - y^2$, (*c*) $iz^2 + 2z$

51. Answer Problem 50 for the function $u = x^2 - y^2 - 2xy - 2x + 3y$. *Ans.* (*b*) $x^2 - y^2 + 2xy - 3x - 2y$

52. Verify that the Cauchy-Riemann equations are satisfied for the functions (*a*) e^{z^2}, (*b*) $\cos 2z$, (*c*) $\sinh 4z$.

53. Determine which of the following functions u are harmonic. For each harmonic function find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.
(*a*) $3x^2y + 2x^2 - y^3 - 2y^2$, (*b*) $2xy + 3xy^2 - 2y^3$, (*c*) $xe^x \cos y - ye^x \sin y$, (*d*) $e^{-2xy}\sin(x^2 - y^2)$.
Ans. (*a*) $v = 4xy - x^3 + 3xy^2 + c$, $f(z) = 2z^2 - iz^3 + ic$ (*c*) $ye^x \cos y + xe^x \sin y + c$, $ze^z + ic$
(*b*) Not harmonic (*d*) $-e^{-2xy}\cos(x^2 - y^2) + c$, $-ie^{iz^2} + ic$

54. (*a*) Prove that $\psi = \ln[(x-1)^2 + (y-2)^2]$ is harmonic in every region which does not include the point (1, 2). (*b*) Find a function ϕ such that $\phi + i\psi$ is analytic. (*c*) Express $\phi + i\psi$ as a function of z.
Ans. (*b*) $-2\tan^{-1}\{(y-2)/(x-1)\}$ (*c*) $2i\ln(z-1-2i)$

55. If $\operatorname{Im}\{f'(z)\} = 6x(2y-1)$ and $f(0) = 3-2i$, $f(1) = 6-5i$, find $f(1+i)$. *Ans.* $6+3i$

DIFFERENTIALS

56. If $w = iz^2 - 4z + 3i$, find (*a*) Δw, (*b*) dw, (*c*) $\Delta w - dw$ at the point $z = 2i$.
Ans. (*a*) $-8\,\Delta z + i(\Delta z)^2 = -8\,dz + i(dz)^2$, (*b*) $-8\,dz$, (*c*) $i(dz)^2$

57. Find (*a*) Δw and (*b*) dw if $w = (2z+1)^3$, $z = -i$, $\Delta z = 1+i$. *Ans.* (*a*) $38-2i$, (*b*) $6-42i$

58. If $w = 3iz^2 + 2z + 1 - 3i$, find (*a*) Δw, (*b*) dw, (*c*) $\Delta w/\Delta z$, (*d*) dw/dz where $z = i$.
Ans. (*a*) $-4\Delta z + 3i(\Delta z)^2$, (*b*) $-4\,dz$, (*c*) $-4 + 3i\,\Delta z$, (*d*) -4

59. (*a*) If $w = \sin z$, show that $\dfrac{\Delta w}{\Delta z} = (\cos z)\left(\dfrac{\sin \Delta z}{\Delta z}\right) - 2\sin z\left\{\dfrac{\sin^2(\Delta z/2)}{\Delta z}\right\}$.

(*b*) Assuming $\lim\limits_{\Delta z \to 0} \dfrac{\sin \Delta z}{\Delta z} = 1$, prove that $\dfrac{dw}{dz} = \cos z$.

(*c*) Show that $dw = (\cos z)\,dz$.

60. (*a*) If $w = \ln z$, show that if $\Delta z/z = \zeta$, $\dfrac{\Delta w}{\Delta z} = \dfrac{1}{z}\ln\{(1+\zeta)^{1/\zeta}\}$.

(*b*) Assuming $\lim\limits_{\zeta \to 0}(1+\zeta)^{1/\zeta} = e = 2.71828\ldots$, prove that $\dfrac{dw}{dz} = \dfrac{1}{z}$.

(*c*) Show that $d(\ln z) = dz/z$.

61. Prove that (*a*) $d\{f(z)\,g(z)\} = \{f(z)\,g'(z) + g(z)\,f'(z)\}\,dz$
(*b*) $d\{f(z)/g(z)\} = \{g(z)\,f'(z) - f(z)\,g'(z)\}\,dz/\{g(z)\}^2$
giving restrictions on $f(z)$ and $g(z)$.

DIFFERENTIATION RULES. DERIVATIVES OF ELEMENTARY FUNCTIONS.

62. Prove that if $f(z)$ and $g(z)$ are analytic in a region $\mathcal{R}$, then
(*a*) $\dfrac{d}{dz}\{2i\,f(z) - (1+i)\,g(z)\} = 2i\,f'(z) - (1+i)\,g'(z)$, (*b*) $\dfrac{d}{dz}\{f(z)\}^2 = 2\,f(z)\,f'(z)$, (*c*) $\dfrac{d}{dz}\{f(z)\}^{-1} = -\{f(z)\}^{-2}\,f'(z)$.

63. Using differentiation rules, find the derivatives of each of the following functions: (*a*) $(1+4i)z^2 - 3z - 2$, (*b*) $(2z+3i)(z-i)$, (*c*) $(2z-i)/(z+2i)$, (*d*) $(2iz+1)^2$, (*e*) $(iz-1)^{-3}$.
Ans. (*a*) $(2+8i)z - 3$, (*b*) $4z + i$, (*c*) $5i/(z+2i)^2$, (*d*) $4i - 8z$, (*e*) $-3i(iz-1)^{-4}$

64. Find the derivatives of each of the following at the indicated points:
(*a*) $(z+2i)(i-z)/(2z-1)$, $z = i$. (*b*) $\{z + (z^2+1)^2\}^2$, $z = 1+i$.
Ans. (*a*) $-6/5 + 3i/5$, (*b*) $-108 - 78i$

65. Prove that (*a*) $\dfrac{d}{dz}\sec z = \sec z \tan z$, (*b*) $\dfrac{d}{dz}\cot z = -\csc^2 z$.

66. Prove that (*a*) $\dfrac{d}{dz}(z^2+1)^{1/2} = \dfrac{z}{(z^2+1)^{1/2}}$, (*b*) $\dfrac{d}{dz}\ln(z^2+2z+2) = \dfrac{2z+2}{z^2+2z+2}$ indicating restrictions if any.