Supplementary Problems

CAUCHY'S INTEGRAL FORMULAS

30. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$ if C is (*a*) the circle $|z|=3$, (*b*) the circle $|z|=1$. *Ans.* (*a*) e^2, (*b*) 0

31. Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z|=5$. *Ans.* $2\pi i$

32. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ if C is (*a*) the circle $|z-1| = 4$, (*b*) the ellipse $|z-2|+|z+2| = 6$.
Ans. (*a*) $-2\pi i$, (*b*) 0

33. Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1}\,dz$ around a rectangle with vertices at: (*a*) $2\pm i$, $-2\pm i$; (*b*) $-i$, $2-i$, $2+i$, i.
Ans. (*a*) 0, (*b*) $-\frac{1}{2}$

34. Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$ if $t>0$ and C is the circle $|z|=3$.

35. Evaluate $\oint_C \frac{e^{iz}}{z^3}\,dz$ where C is the circle $|z|=2$. *Ans.* $-\pi i$

36. Prove that $f'''(a) = \frac{3!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^4}$ if C is a simple closed curve enclosing $z=a$ and $f(z)$ is analytic inside and on C.

37. Prove Cauchy's integral formulas for all positive integral values of n. [*Hint*: Use mathematical induction.]

38. Find the value of (*a*) $\oint_C \frac{\sin^6 z}{z-\pi/6}\,dz$, (*b*) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ if C is the circle $|z|=1$.
Ans. (*a*) $\pi i/32$, (*b*) $21\pi i/16$

39. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $t>0$ and C is the circle $|z|=3$. *Ans.* $\frac{1}{2}(\sin t - t\cos t)$

40. Prove Cauchy's integral formulas for the multiply-connected region of Fig. 4-26, Page 115.

MORERA'S THEOREM

41. (*a*) Determine whether $G(z) = \int_1^z \frac{d\zeta}{\zeta}$ is independent of the path joining 1 and z.
(*b*) Discuss the relationship of your answer to part (*a*) with Morera's theorem.

42. Does Morera's theorem apply in a multiply-connected region? Justify your answer.

43. (*a*) If $P(x,y)$ and $Q(x,y)$ are conjugate harmonic functions and C is any simple closed curve, prove that $\oint_C P\,dx + Q\,dy = 0$.
(*b*) If for all simple closed curves C in a region $\mathcal{R}$, $\oint_C P\,dx + Q\,dy = 0$, is it true that P and Q are conjugate harmonic functions, i.e. is the converse of (*a*) true? Justify your conclusion.

CAUCHY'S INEQUALITY

44. (*a*) Use Cauchy's inequality to obtain estimates for the derivatives of $\sin z$ at $z=0$ and (*b*) determine how good these estimates are.

45. (*a*) Show that if $f(z) = 1/(1-z)$, then $f^{(n)}(z) = n!/(1-z)^{n+1}$.

(*b*) Use (*a*) to show that the Cauchy inequality is "best possible", i.e. the estimate of growth of the nth derivative cannot be improved for *all* functions.

46. Prove that the equality in cauchy's inequality (*3*), Page 118, holds in the case $n = m$ if and only if $f(z) = kM(z-a)^m/r^m$, where $|k| = 1$.

47. Discuss Cauchy's inequality for the function $f(z) = e^{-1/z^2}$ in the neighborhood of $z = 0$.

LIOUVILLE'S THEOREM

48. The function of a real variable defined by $f(x) = \sin x$ is (*a*) analytic everywhere and (*b*) bounded, i.e. $|\sin x| \leqq 1$ for all x but it is certainly not a constant. Does this contradict Liouville's theorem? Explain.

49. A non-constant function $F(z)$ is such that $F(z+a) = F(z)$, $F(z+bi) = F(z)$ where $a > 0$ and $b > 0$ are given constants. Prove that $F(z)$ cannot be analytic in the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$.

FUNDAMENTAL THEOREM OF ALGEBRA

50. (*a*) Carry out the details of proof of the fundamental theorem of algebra to show that the particular function $f(z) = z^4 - z^2 - 2z + 2$ has exactly four zeros. (*b*) Determine the zeros of $f(z)$.
Ans. (*b*) $1, 1, -1 \pm i$

51. Determine all the roots of the equations (*a*) $z^3 - 3z + 4i = 0$, (*b*) $z^4 + z^2 + 1 = 0$.
Ans. (*a*) i, $\frac{1}{2}(-i \pm \sqrt{15})$, (*b*) $\frac{1}{2}(-1 \pm \sqrt{3}\,i)$, $\frac{1}{2}(1 \pm \sqrt{3}\,i)$

GAUSS' MEAN VALUE THEOREM

52. Evaluate $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin^2(\pi/6 + 2e^{i\theta})\,d\theta$ *Ans.* 1/4

53. Show that the mean value of any harmonic function over a circle is equal to the value of the function at the center.

54. Find the mean value of $x^2 - y^2 + 2y$ over the circle $|z - 5 + 2i| = 3$. *Ans.* 17

55. Prove that $\displaystyle\int_0^{\pi} \ln \sin\theta\,d\theta = -\pi \ln 2$. [*Hint.* Consider $f(z) = \ln(1+z)$.]

MAXIMUM MODULUS THEOREM

56. Find the maximum of $|f(z)|$ in $|z| \leqq 1$ for the functions $f(z)$ given by (*a*) $z^2 - 3z + 2$, (*b*) $z^4 + z^2 + 1$, (*c*) $\cos 3z$, (*d*) $(2z+1)/(2z-1)$.

57. (*a*) If $f(z)$ is analytic inside and on the simple closed curve C enclosing $z = a$, prove that

$$\{f(a)\}^n = \frac{1}{2\pi i}\oint_C \frac{\{f(z)\}^n}{z-a}\,dz \qquad n = 0, 1, 2, \ldots$$

(*b*) Use (*a*) to prove that $|f(a)|^n \leqq M^n/2\pi D$ where D is the minimum distance from a to the curve C and M is the maximum value of $|f(z)|$ on C.

(*c*) By taking the nth root of both sides of the inequality in (*b*) and letting $n \to \infty$, prove the maximum modulus theorem.

58. Let $U(x,y)$ be harmonic inside and on a simple closed curve C. Prove that the (*a*) maximum and (*b*) minimum values of $U(x,y)$ are attained on C. Are there other restrictions on $U(x,y)$?

59. Verify Problem 58 for the functions (*a*) $x^2 - y^2$ and (*b*) $x^3 - 3xy^2$ if C is the circle $|z| = 1$.

60. Is the maximum modulus theorem valid for multiply-connected regions? Justify your answer.