57. Investigate the convergence of:

(a) $\sum_{n=1}^{\infty} \frac{1}{n+|z|}$, (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+|z|}$, (c) $\sum_{n=1}^{\infty} \frac{1}{n^2+|z|}$, (d) $\sum_{n=1}^{\infty} \frac{1}{n^2+z}$.

Ans. (a) Diverges for all finite z. (b) Converges for all z. (c) Converges for all z. (d) Converges for all z except $z = -n^2$, $n = 1, 2, 3, \ldots$.

58. Investigate the convergence of $\sum_{n=0}^{\infty} \frac{ne^{n\pi i/4}}{e^n - 1}$. *Ans.* Conv.

59. Find the region of convergence of (a) $\sum_{n=0}^{\infty} \frac{(z+i)^n}{(n+1)(n+2)}$, (b) $\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 3^n}\left(\frac{z+1}{z-1}\right)^n$, (c) $\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!}$.

Ans. (a) $|z+i| \leqq 1$, (b) $|(z+1)/(z-1)| \leqq 3$, (c) $|z| < \infty$

60. Investigate the region of absolute convergence of $\sum_{n=1}^{\infty} \frac{n(-1)^n(z-i)^n}{4^n(n^2+1)^{5/2}}$.

Ans. Conv. abs. for $|z-i| \leqq 4$.

61. Find the region of convergence of $\sum_{n=0}^{\infty} \frac{e^{2\pi i n z}}{(n+1)^{3/2}}$.

Ans. Converges if $\operatorname{Im} z \geqq 0$.

62. Prove that the series $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ diverges although the nth term approaches zero.

63. Let N be a positive integer and suppose that for all $n > N$, $|u_n| > 1/(n \ln n)$. Prove that $\sum_{n=1}^{\infty} u_n$ diverges.

64. Establish the validity of the (a) nth root test [Theorem 12], (b) integral test [Theorem 13], on Page 141.

65. Find the interval of convergence of $1 + 2z + z^2 + 2z^3 + z^4 + 2z^5 + \cdots$. *Ans.* $|z| < 1$

66. Prove Raabe's test (Theorem 14) on Page 141.

67. Test for convergence: (a) $\frac{1}{2\ln^2 2} + \frac{1}{3\ln^2 3} + \frac{1}{4\ln^2 4} + \cdots$, (b) $\frac{1}{5} + \frac{1\cdot 4}{5\cdot 8} + \frac{1\cdot 4\cdot 7}{5\cdot 8\cdot 11} + \cdots$, (c) $\frac{2}{5} + \frac{2\cdot 7}{5\cdot 10} + \frac{2\cdot 7\cdot 12}{5\cdot 10\cdot 15} + \cdots$, (d) $\frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} + \cdots$.

Ans. (a) conv., (b) conv., (c) div., (d) div.

THEOREMS ON UNIFORM CONVERGENCE AND POWER SERIES

→ **68.** Determine the regions in which each of the following series is uniformly convergent:

all parts (a) $\sum_{n=1}^{\infty} \frac{z^n}{3^n+1}$, ** (b) $\sum_{n=1}^{\infty} \frac{(z-i)^{2n}}{n^2}$, ** (c) $\sum_{n=1}^{\infty} \frac{1}{(n+1)z^n}$, (d) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+|z|^2}$.

Ans. (a) $|z| \leqq R$ where $R < 3$. (b) $|z-i| \leqq 1$. (c) $|z| \geqq R$ where $R > 1$. (d) All z

69. Prove Theorem 20, Page 142.

70. State and prove theorems for sequences analogous to Theorems 18, 19 and 20, Page 142, for series.

→ **71.** (a) By differentiating both sides of the identity

both parts

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$$

find the sum of the series $\sum_{n=1}^{\infty} nz^n$ for $|z| < 1$. Justify all steps.

** (b) Find the sum of the series $\sum_{n=1}^{\infty} n^2 z^n$ for $|z| < 1$.

Ans. (a) $z/(1-z)^2$ [compare Problem 44], (b) $z(1+z)/(1-z)^3$

72. Let z be real and such that $0 \leqq z \leqq 1$, and let $u_n(z) = nze^{-nz^2}$. (*a*) Find $\lim\limits_{n\to\infty} \int_0^1 u_n(z)\,dz$. (*b*) Find $\int_0^1 \left\{\lim\limits_{n\to\infty} u_n(z)\right\} dz$. (*c*) Explain why the answers to (*a*) and (*b*) are not equal. [See Problem 53.]

Ans. (*a*) 1/2, (*b*) 0

73. Prove Abel's theorem [Theorem 24, Page 142].

74. (*a*) Prove that $\dfrac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$ for $|z| < 1$.

(*b*) If we choose that branch of $f(z) = \tan^{-1} z$ such that $f(0) = 0$, use (*a*) to prove that

$$\tan^{-1} z = \int_0^z \frac{dz}{1+z^2} = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

(*c*) Prove that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$.

75. Prove Theorem 25, Page 142.

76. (*a*) Determine $Y(z) = \sum_{n=0}^{\infty} a_n z^n$ such that for all z in $|z| \leqq 1$, $Y'(z) = Y(z)$, $Y(0) = 1$. State all theorems used and verify that the result obtained is a solution.

(*b*) Is the result obtained in (*a*) valid outside of $|z| \leqq 1$? Justify your answer.

(*c*) Show that $Y(z) = e^z$ satisfies the differential equation and conditions in (*a*).

(*d*) Can we identify the series in (*a*) with e^z? Explain.

Ans. (*a*) $Y(z) = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$

77. (*a*) Use series methods on the differential equation $Y''(z) + Y(z) = 0$, $Y(0) = 0$, $Y'(0) = 1$ to obtain the series expansion

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

(*b*) How could you obtain a corresponding series for $\cos z$?

TAYLOR'S THEOREM

78. Expand each of the following functions in a Taylor series about the indicated point and determine the region of convergence in each case.

(*a*) e^{-z}; $z = 0$ (*b*) $\cos z$; $z = \pi/2$ (*c*) $1/(1+z)$; $z = 1$ (*d*) $z^3 - 3z^2 + 4z - 2$; $z = 2$ (*e*) ze^{2z}; $z = -1$

79. If each of the following functions were expanded into a Taylor series about the indicated points, what would be the region of convergence? Do not perform the expansion.

(*a*) $\sin z/(z^2+4)$; $z = 0$ (*c*) $(z+3)/(z-1)(z-4)$; $z = 2$ (*e*) $e^z/z(z-1)$; $z = 4i$ (*g*) $\sec \pi z$; $z = 1$

(*b*) $z/(e^z+1)$; $z = 0$ (*d*) $e^{-z^2}\sinh(z+2)$; $z = 0$ (*f*) $z \coth 2z$; $z = 0$

Ans. (*a*) $|z| < 2$, (*b*) $|z| < \pi$, (*c*) $|z-2| < 1$, (*d*) $|z| < \infty$, (*e*) $|z-4i| < 4$, (*f*) $|z| < \pi/2$, (*g*) $|z-1| < 1/2$

80. Verify the expansions 1, 2, 3 for e^z, $\sin z$ and $\cos z$ on Page 143.

81. Show that $\sin z^2 = z^2 - \dfrac{z^6}{3!} + \dfrac{z^{10}}{5!} - \dfrac{z^{14}}{7!} + \cdots$, $|z| < \infty$.

82. Prove that $\tan^{-1} z = z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \dfrac{z^7}{7} + \cdots$, $|z| < 1$.

83. Show that

(*a*) $\tan z = z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots$, $|z| < \pi/2$

(*b*) $\sec z = 1 + \dfrac{z^2}{2} + \dfrac{5z^4}{24} + \cdots$, $|z| < \pi/2$

(*c*) $\csc z = \dfrac{1}{z} + \dfrac{z}{6} + \dfrac{7z^3}{360} + \cdots$, $0 < |z| < \pi$