84. By replacing z by iz in the expansion of Problem 82, obtain the result in Problem 23(*c*) on Page 155.

85. How would you obtain series for (*a*) $\tanh z$, (*b*) $\operatorname{sech} z$, (*c*) $\operatorname{csch} z$ from the series in Problem 83?

86. Prove the uniqueness of the Taylor series expansion of $f(z)$ about $z = a$.
[*Hint.* Assume $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} d_n(z-a)^n$ and show that $c_n = d_n$, $n = 0, 1, 2, 3, \ldots$.]

87. Prove the binomial Theorem 6 on Page 143.

88. If we choose that branch of $\sqrt{1+z^3}$ having the value 1 for $z = 0$, show that

$$\frac{1}{\sqrt{1+z^3}} = 1 - \frac{1}{2}z^3 + \frac{1\cdot 3}{2\cdot 4}z^6 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^9 + \cdots \qquad |z| < 1$$

89. (*a*) Choosing that branch of $\sin^{-1} z$ having the value zero for $z = 0$, show that

$$\sin^{-1} z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^7}{7} + \cdots \qquad |z| < 1$$

(*b*) Prove that the result in (*a*) is valid for $z = i$.

90. (*a*) Expand $f(z) = \ln(3 - iz)$ in powers of $z - 2i$, choosing that branch of the logarithm for which $f(0) = \ln 3$, and (*b*) determine the region of convergence.

Ans. (*a*) $\ln 5 - \frac{i(z-2i)}{5} + \frac{(z-2i)^2}{2\cdot 5^2} + \frac{i(z-2i)^3}{3\cdot 5^3} - \frac{(z-2i)^4}{4\cdot 5^4} - \cdots$ (*b*) $|z - 2i| < 5$

LAURENT'S THEOREM

91. Expand $f(z) = 1/(z-3)$ in a Laurent series valid for (*a*) $|z| < 3$, (*b*) $|z| > 3$.

Ans. (*a*) $-\frac{1}{3} - \frac{1}{9}z - \frac{1}{27}z^2 - \frac{1}{81}z^3 - \cdots$ (*b*) $z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \cdots$

92. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for:

(*a*) $|z| < 1$, (*b*) $1 < |z| < 2$, (*c*) $|z| > 2$, (*d*) $|z-1| > 1$, (*e*) $0 < |z-2| < 1$.

Ans. (*a*) $-\frac{1}{2}z - \frac{3}{4}z^2 - \frac{7}{8}z^3 - \frac{15}{16}z^4 - \cdots$ (*b*) $\cdots + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \cdots$

(*c*) $-\frac{1}{2} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \cdots$ (*d*) $-(z-1)^{-1} - 2(z-1)^{-2} - 2(z-1)^{-3} - \cdots$

(*e*) $1 - 2(z-2)^{-1} - (z-2) + (z-2)^2 - (z-2)^3 + (z-2)^4 - \cdots$

93. Expand $f(z) = 1/z(z-2)$ in a Laurent series valid for (*a*) $0 < |z| < 2$, (*b*) $|z| > 2$.

94. Find an expansion of $f(z) = z/(z^2+1)$ valid for $|z-3| > 2$.

95. Expand $f(z) = 1/(z-2)^2$ in a Laurent series valid for (*a*) $|z| < 2$, (*b*) $|z| > 2$

96. Expand each of the following functions in a Laurent series about $z = 0$, naming the type of singularity in each case.

(*a*) $(1 - \cos z)/z$, (*b*) e^{z^2}/z^3, (*c*) $z^{-1}\cosh z^{-1}$, (*d*) $z^2 e^{-z^4}$, (*e*) $z \sinh\sqrt{z}$.

Ans. (*a*) $\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots$; removable singularity

(*b*) $\frac{1}{z^3} + \frac{1}{z} + \frac{z}{2!} + \frac{z^3}{3!} + \frac{z^5}{4!} + \frac{z^7}{5!} + \cdots$; pole of order 3

(*c*) $\frac{1}{z} - \frac{1}{2!\,z^3} + \frac{1}{4!\,z^5} - \cdots$; essential singularity

(*d*) $z^2 - z^6 + \frac{z^{10}}{2!} - \frac{z^{14}}{3!} + \cdots$; ordinary point

(*e*) $z^{3/2} + \frac{z^{5/2}}{3!} + \frac{z^{7/2}}{5!} + \frac{z^{9/2}}{7!} + \cdots$; branch point

97. Show that if $\tan z$ is expanded into a Laurent series about $z = \pi/2$, (*a*) the principal part is $-1/(z - \pi/2)$, (*b*) the series converges for $0 < |z - \pi/2| < \pi/2$, (*c*) $z = \pi/2$ is a simple pole.

98. Determine and classify all the singularities of the functions:
(*a*) $1/(2\sin z - 1)^2$, (*b*) $z/(e^{1/z} - 1)$, (*c*) $\cos(z^2 + z^{-2})$, (*d*) $\tan^{-1}(z^2 + 2z + 2)$, (*e*) $z/(e^z - 1)$.

Ans. (*a*) $\pi/6 + 2m\pi$, $(2m+1)\pi - \pi/6$, $m = 0, \pm 1, \pm 2, \ldots$; poles of order 2
(*b*) $i/2m\pi$, $m = \pm 1, \pm 2, \ldots$; simple poles, $z = 0$; essential singularity, $z = \infty$; pole of order 2
(*c*) $z = 0, \infty$; essential singularities (*d*) $z = -1 \pm i$; branch points
(*e*) $z = 2m\pi i$, $m = \pm 1, \pm 2, \ldots$; simple poles, $z = 0$; removable singularity, $z = \infty$; essential singularity

99. (*a*) Expand $f(z) = e^{z/(z-2)}$ in a Laurent series about $z = 2$ and (*b*) determine the region of convergence of this series. (*c*) Classify the singularities of $f(z)$.

Ans. (*a*) $e\left\{1 + 2(z-2)^{-1} + \dfrac{2^2(z-2)^{-2}}{2!} + \dfrac{2^3(z-2)^{-3}}{3!} + \cdots\right\}$ (*b*) $|z-2| > 0$ (*c*) $z = 2$; essential singularity, $z = \infty$; removable singularity

100. Establish the result (*7*), Page 144, for the coefficients in a Laurent series.

101. Prove that the only singularities of a rational function are poles.

102. Prove the converse of Problem 101, i.e. if the only singularities of a function are poles, the function must be rational.

LAGRANGE'S EXPANSION

103. Show that the root of the equation $z = 1 + \zeta z^p$, which is equal to 1 when $\zeta = 0$, is given by

$$z = 1 + \zeta + \frac{2p}{2!}\zeta^2 + \frac{(3p)(3p-1)}{3!}\zeta^3 + \frac{(4p)(4p-1)(4p-2)}{4!}\zeta^4 + \cdots$$

104. Calculate the root in Problem 103 if $p = 1/2$ and $\zeta = 1$, (*a*) by series and (*b*) exactly, and compare the two answers. *Ans.* 2.62 to two decimal accuracy

105. By considering the equation $z = \alpha + \frac{1}{2}\zeta(z^2 - 1)$, show that

$$\frac{1}{\sqrt{1 - 2\alpha\zeta + \zeta^2}} = 1 + \sum_{n=1}^{\infty} \frac{\zeta^n}{2^n\, n!} \frac{d^n}{d\alpha^n}(\alpha^2 - 1)^n$$

106. Show how Lagrange's expansion can be used to solve Kepler's problem of determining that root of $z = a + \zeta \sin z$ for which $z = a$ when $\zeta = 0$.

107. Prove the Lagrange expansion (*12*) on Page 145.

ANALYTIC CONTINUATION

108. (*a*) Prove that $F_2(z) = \dfrac{1}{1+i}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z+i}{1+i}\right)^n$ is an analytic continuation of $F_1(z) = \displaystyle\sum_{n=0}^{\infty} z^n$, showing graphically the regions of convergence of the series.

(*b*) Determine the function represented by all analytic continuations of $F_1(z)$. *Ans.* (*b*) $1/(1-z)$

109. Let $F_1(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^{n+1}}{3^n}$. (*a*) Find an analytic continuation of $F_1(z)$ which converges for $z = 3 - 4i$.
(*b*) Determine the value of the analytic continuation in (*a*) for $z = 3 - 4i$. *Ans.* (*b*) $-3 - \frac{9}{4}i$

110. Prove that the series

$$z^{1!} + z^{2!} + z^{3!} + \cdots$$

has the natural boundary $|z| = 1$.

MISCELLANEOUS PROBLEMS

111. (*a*) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if the constant $p \leqq 1$.

(*b*) Prove that if p is complex the series in (*a*) converges if $\text{Re}\{p\} > 1$.

(*c*) Investigate the convergence or divergence of the series in (*a*) if $\text{Re}\{p\} \leqq 1$.