Q1. Let \((X, T)\) be a Hausdorff space. Prove that each single point set \(\{x\}\) is closed.

Q2. Let \((X, T)\) be a compact Hausdorff space. Let \(F\) be a closed subset and let \(x_0 \in F^c\). Prove that there are disjoint open sets \(U\) and \(V\) such that \(x_0 \in U\) and \(F \subseteq V\). (Hint: start by applying the Hausdorff condition to \(x_0\) and an arbitrary point in \(F\))

Q3. Use Q2. to prove that if \(E\) and \(F\) are disjoint closed sets in a compact Hausdorff space \((X, T)\) then there are disjoint open sets \(U\) and \(V\) such that \(E \subseteq U\) and \(F \subseteq V\).

Q4. Let \((x_\alpha)_{\alpha \in A}\) be a net in \((X, T)\) converging to a point \(x\). Prove that a subnet \((y_\beta)_{\beta \in B}\) also converges to \(x\).

Q5. Let \(S\) be a set with the discrete topology. Prove that the compact subsets are the finite ones.

Q6. Let \(E\) be the Euclidean topology on \(\mathbb{R}\), let \(Z\) be the integers and let \(S_n = \{0, \pm 1, \ldots, \pm n\}\). Let \(E_n\) be the collection of sets of the form \(U \cup V\) where \(U \in E\) and \(V\) is an arbitrary subset of \(S_n\). Prove that \(E_n\) is a topology. Prove that \(Z\) is open in any topology containing the \(E_n\)'s, but is not in \(E_n\) for any \(n\), and thus that \(\bigcup_{n \geq 1} E_n\) is not a topology.

Q7. In the plane, identify \((\mathbb{R}, E)\) with the \(x\)-axis and let \(\omega\) be the point \((0,1)\) on the \(y\)-axis. Define a topology \(T\) on \(\mathbb{R} \cup \{\omega\}\) by declaring a set to be open if it is either an \(E\)-open subset of \(\mathbb{R}\) or the complement of an \(E\)-compact subset \(K\) of \(\mathbb{R}\). Prove that \(T\) is a topology for which \(\mathbb{R} \cup \{\omega\}\) is compact. Define \(f: [0,1] \to \mathbb{R} \cup \{\omega\}\) by \(f(x) = x/(1-x)\) for \(0 \leq x < 1\) and \(f(1) = \omega\). Prove that this is a continuous path from 0 to \(\omega\).
447 #2 Solutions

Q1. For each \( y \in \{x \times y \} \) choose open sets \( U_y, V_y \) s.t. \( x \in U_y \), \( y \in V_y \). Then \( U \cup V_y = \{x \times y \} \) \( y \in \{x \times y \} \)

which is then open, so \( \{x \times y\} \) is closed.

Q2. As in Q1, choose disjoint sets

\( x_0 \in U_y, y \in V_y \) for each \( y \in F \). Then \( V_y \)'s cover \( F \), so by compactness a finite number \( V_{y_1}, \ldots, V_{y_n} \) covers \( F \). Let

\[
U = \bigcap_{i=1}^{n} U_{y_i}, \quad V = \bigcup_{i=1}^{n} V_{y_i}.
\]

Then \( U \cap V = \emptyset \), \( x_0 \in U \) and \( F \subseteq V \).

Q3. Using Q2, for each \( x \in E \) get disjoint open sets \( x \in U_x, F \subseteq V_x \). A finite no

\( U_{x_1}, \ldots, U_{x_n} \) cover \( E \) by compactness, so let

\[
U = \bigcap_{x} U_{x}, \quad V = \bigcup_{x} V_{x},
\]

open and disjoint with \( E \subseteq U, F \subseteq V \).
C4. Let U be an open set containing x. Then there exists \( \varepsilon_0 \) such that \( x \in U \) for \( \varepsilon \geq \varepsilon_0 \).

Choose \( \varepsilon_0 \) so that \( \{ y_\beta : \beta \geq \beta_0 \} \subseteq \{ x_\varepsilon : \varepsilon \geq \varepsilon_0 \} \)

Then \( y_\beta \in U \) for \( \beta \geq \beta_0 \). Thus \( y_\beta \to x \).

6S. If \( S = \{ x_1, \ldots, x_n \} \) and is covered by open sets \( U_i \), choose \( U_i \) containing \( x_i \) for \( 1 \leq i \leq n \). Then \( S \subseteq \bigcup U_i \), and \( S \) is compact (true for any topology).

Conversely, if \( S \) is compact, cover by the open sets \( \{ \{ x \} : x \in S \} \). A finite no. cover \( S \), so \( S \) has a finite no. of points.
Q6 \( \phi, R \in E \) and hence in \( E_n \).

If \( W = U_1 \cup U_2 \) with \( U \in E_n \), \( V \in E \)

Then \( U U_1, U U_2 \in E_n \) and \( U V \in E \), so

\[ U W_2 \in E_n. \] If \( W_i = U_i \cup V_i, i = 1, 2 \)

Then \( W_1 \cap W_2 = (U_1 \cap U_2) \cup (U_1 \cap V_2) \cup (V_1 \cap U_2) \cup (V_1 \cap V_2) \),

#3 is in \( E \) and #1, 2, 4 are subsets of \( S_n \)

so \( W_1 \cap W_2 \in E_n \), so \( E_n \) is a topology.

Since \( Z = \bigcup S_n \) so \( Z \) is open in any topology containing \( E_n \) for all \( n \). But

\[ Z \notin E_n \] so \( \bigcup E_n \) is not a topology.

Q7. If \( U_1 \) are open (\( \phi \) there are 2 cases.

1. \( U_1 \subseteq R \) for all \( n \).

Then \( U U_1 \subseteq R \), so open in \( T \).

2. There is a \( U_1 \) containing \( \phi \).

Then \( U_1 \subseteq \text{compact in } R \), so

\[ (U U_1)^c \subseteq U_1 \subseteq \text{compact} \]. Thus

\( \phi \) is closed under arbitrary unions.
Consider \( U_1 \cup U_2 \) 

1. If \( U_1 \cap U_2 \neq \emptyset \) then \( U_1 \cup U_2 \) is \( T \)-open.

2. If \( w \in U_1 \cup U_2 \) then \( U_1^c \) and \( U_2^c \) are compact subsets of \( IR \) and so is \( U_1^c \cup U_2^c \).

So \( U_1 \cup U_2 \) is open.

3. If \( w \in U_1 \) or \( U_2 \), then \( U_1 \cup U_2 \) is a subset of \( IR \). Since \( U_1^c \subseteq IR \) is compact \( IR \setminus U_1^c \) is open.

Then \( U_1 \cup U_2 = (IR \setminus U_1^c) \cup U_2 \), open in \( IR \).

Thus \( T \) is a topology.

Give an open cover \( U_x \) of \( IR \setminus \{ 0 \} \)

at least one, \( U_{x_i} \) contains \( 0 \) so its complement is compact so covered by a finite no. \( U_{x_1}, \ldots, U_{x_n} \). Then \( U_{x_1}, \ldots, U_{x_n} \) cover \( IR \cup \{ 0 \} \), which is then compact.
Define $f : [0,1] \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$f(x) = \frac{x}{1-x} \quad \text{for} \quad 0 \leq x < 1, \quad f(1) = \infty.$$ 

Consider a sequence $x_n \rightarrow x \in [0,1]$. 

If $x < 1$ then for $n$ large enough $x_n < 1$

so $f(x_n) = \frac{x_n}{1-x_n} \rightarrow \frac{x}{1-x} = f(x)$.

If $x = 1$ then $f(x_n) \rightarrow \infty$. Given an open set $U$ containing $x$, $U^c$ is a compact subset of $\mathbb{R}$, so for $n$ large enough $f(x_n)$ is outside $U^c$, and then lies in $U$. Thus $f(x_n) \rightarrow \infty$ since $U$ was arbitrary.