ANGULAR VALUE DISTRIBUTION OF POWER SERIES WITH GAPS

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1. Introduction

Let

$$(1.1)\qquad f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$

be an integral function with gaps, i.e. such that most of the a_n are zero, in a certain sense. We shall prove in this case that $f(z)$ assumes all finite values in every angle with a density that is proportional to the size of the angle.

Suppose that $\varphi(n)$ is the number of non-zero terms among the coefficients a_1 to a_n. Then Pólya ([**12**]) proved that if $f(z)$ has Fabry gaps, i.e. if

$$(1.2)\qquad \frac{\varphi(n)}{n} \to 0 \quad \text{as} \quad n \to \infty,$$

and if $f(z)$ has infinite order, then every line is a line of Julia. Thus $f(z)$ assumes in every angle every value with at most one exception infinitely often. This result has recently been extended by Anderson and Clunie ([**1**]) to functions of finite positive order. The case of zero order has remained open.

Biernacki ([**3**]) has shown that if the indices n_k, for which $a_n \neq 0$ satisfy the gap condition

$$(1.3)\qquad n_{k+1} - n_k > n_{k+1}{}^{\frac{1}{2}+\delta} \quad (k > k_0)$$

for some $\delta > 0$, then $f(z)$ assumes all values in every angle. We shall prove a more precise result under a weaker hypothesis than (1.3), which reduces to (1.2) if the lower order of $f(z)$ is finite. Thus in particular (1.2) always implies that every line is a line of Julia.

2. Notation and statement of results

We shall assume that $f(z)$ given by (1.1) is regular in $|z| < \rho_0$, where $0 < \rho_0 \leqslant \infty$, and write

$$N(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta,$$

which corresponds to the value $N(r, 0)$ in Nevanlinna theory. We also put

$$L(r,f) = \inf_{|z|=r} |f(z)|, \quad M(r,f) = \sup_{|z|=r} |f(z)|, \quad 0 < r < \rho_0.$$

Suppose that $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$, and write $n(r, \theta_1, \theta_2, a)$ for the number of roots of the equation $f(z) = a$ in the open sector

$$S(r, \theta_1, \theta_2) = \{z \mid 0 < |z| < r, \quad \theta_1 < \arg z < \theta_2\}. \tag{2.1}$$

We also put

$$N(r, \theta_1, \theta_2, a) = \int_0^r n(t, \theta_1, \theta_2, a) \frac{dt}{t}. \tag{2.2}$$

Then by Jensen's formula, $N(r,f) = N(r, 0, 2\pi, 0)$.

We recall briefly the definitions of density and logarithmic density for a measurable set E on the positive real axis.

Let $E(a, b)$ denote the part of E in the interval (a, b). Then the lower density $\underline{\text{dens}}\, E$ and upper density $\overline{\text{dens}}\, E$ are defined respectively by

$$\underline{\overline{\text{dens}}}\, E = \underline{\overline{\lim}}_{r\to\infty} \frac{1}{r} \int_{E(0,r)} dt, \tag{2.3}$$

and similarly the upper logarithmic density $\overline{\log \text{dens}}\ E$ and lower logarithmic density $\underline{\log \text{dens}}\ E$ are defined by

$$\underline{\overline{\log \text{dens}}}\, E = \underline{\overline{\lim}}_{r\to\infty} \frac{1}{\log r} \int_{E(1,r)} \frac{dt}{t}. \tag{2.4}$$

If upper and lower density (logarithmic density) are equal, their common value is called the density (logarithmic density) of E.

2.1. Our results will be based on an extension of Jensen's formula. We suppose that $f(z)$ is meromorphic near the segment $z = te^{i\theta}$ $(0 < t < r)$, that $f(0) = 1$ and that $f(z) \neq 0$, on the segment.

Then we can define a unique value of $\arg f(te^{i\theta})$ which is continuous for $0 < t < r$ and reduces to zero when $t = 0$. We denote this value by $v(t, \theta)$ and write

$$V(r, \theta) = \frac{1}{2\pi} \int_0^r v(t, \theta) \frac{dt}{t}. \tag{2.5}$$

We then have the following formula, which appears implicitly in Pfluger's paper [**11**], pp. 50, 59 and more explicitly in Levin's book [**10**], p. 188. We include the proof for completeness.

THEOREM 1. *Suppose that $f(z)$ is meromorphic in the closure of the sector $S(r, \theta_1, \theta_2)$, that $f(0) = 1$, and that $f(z) \neq 0$ on the arms $\arg z = \theta_1, \theta_2$ of S.*

Then

(2.6) $$N(r,\theta_1,\theta_2,0)-N(r,\theta_1,\theta_2,\infty) = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\log|f(re^{i\theta})|\,d\theta+V(r,\theta_1)-V(r,\theta_2).$$

Proof. We apply the argument principle to the sector $S(\rho,\theta_1,\theta_2)$, where $0<\rho<r$ and ρ is such that $f(z)\neq 0,\infty$, for $z=\rho e^{i\theta}$ $(\theta_1<\theta<\theta_2)$. This gives

$$2\pi\{n(\rho,\theta_1,\theta_2,0)-n(\rho,\theta_1,\theta_2,\infty)\}$$
$$=v(\rho,\theta_1)-v(\rho,\theta_2)+\int_{\theta_1}^{\theta_2}\frac{\partial \arg f(\rho e^{i\theta})}{\partial\theta}\,d\theta$$
$$=v(\rho,\theta_1)-v(\rho,\theta_2)+\int_{\theta_1}^{\theta_2}\rho\frac{\partial}{\partial\rho}\log|f(\rho e^{i\theta})|\,d\theta.$$

We integrate this equation from $\rho=\rho_1$ to ρ_2, where $f(z)\neq 0$, ∞, for $\rho_1\leqslant|z|\leqslant\rho_2$, $\theta_1\leqslant\arg f(z)\leqslant\theta_2$, having first divided by ρ. This gives

(2.7) $$N(\rho_2,\theta_1,\theta_2,0)-N(\rho_1,\theta_1,\theta_2,0)-\{N(\rho_2,\theta_1,\theta_2,\infty)-N(\rho_1,\theta_1,\theta_2,\infty)\}$$
$$=\frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\log|f(\rho_2e^{i\theta})|\,d\theta-\int_{\theta_1}^{\theta_2}\log|f(\rho_1e^{i\theta})|\,d\theta$$
$$+V(\rho_2,\theta_1)-V(\rho_1,\theta_1)+V(\rho_1,\theta_2)-V(\rho_2,\theta_2).$$

Since all terms are continuous in ρ_1,ρ_2 this latter equation remains valid if $f(z)$ has a finite number of poles and zeros on $|z|=\rho_1,\rho_2$, but none for $\rho_1<|z|<\rho_2$. Also since an arbitrary interval $[\rho_1,\rho_2]$, for $0<\rho_1<\rho_2<r$, may be split into a finite number of intervals of this latter type, we deduce (2.7) for general ρ_1,ρ_2. Finally, we let ρ_1 tend to zero, in which case all the terms involving ρ_1 tend to zero since $f(0)=1$, and put $\rho_2=r$ in (2.7). This proves Theorem 1.

We now put

(2.8) $$V_0(r,\theta)=\frac{1}{2\pi}\int_0^r|v(t,\theta)|\frac{dt}{t},$$

so that

$$|V(r,\theta)|\leqslant V_0(r,\theta),$$

and

(2.9) $$I_0(r)=\frac{1}{2\pi}\int_0^{2\pi}V_0(r,\theta)\,d\theta.$$

We can now state our fundamental result.

THEOREM 2. *Suppose that $f(z)$ is regular in $|z| < \rho_0$, and $f(0) = 1$. Then we have, for $0 < r < \rho_0$, $\theta_1 < \theta_2 < 2\pi$,*

$$\left|N(r,\theta_1,\theta_2,0) - \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \log|f(re^{i\theta})|\,d\theta\right| < 4\{I_0(r)\log M(r)\}^{\frac{1}{2}}. \tag{2.10}$$

COROLLARY. *If $r \to \rho_0$ through a set of values E, such that*

$$\log L(r,f) \sim \log M(r) \tag{2.11}$$

and

$$I_0(r) = o\{\log M(r)\}, \tag{2.12}$$

then we have, uniformly for $\theta_1 < \theta_2 < \theta_1 + 2\pi$,

$$N(r,\theta_1,\theta_2,0) = \left\{\frac{\theta_2-\theta_1}{2\pi} + o(1)\right\}\log M(r) \tag{2.13}$$

as $r \to \rho_0$ on E.

If a is any complex number we put

$$f_a(z) = \frac{f(z)-a}{1-a} \quad (a \neq 1), \tag{2.14}$$

$$f_1(z) = cz^{-p}\{f(z)-1\},$$

where c, p are chosen so that $f_1(0) = 1$. Then evidently (2.11) will hold simultaneously for $f_a(z)$ and $f(z)$ and the same will apply to our estimates for $I_0(r)$. Thus in effect we shall be able to apply the conclusions of Theorem 2, Corollary, simultaneously to all the functions $f_a(z)$ as $r \to \rho_0$ through suitable values. If $\rho_0 = \infty$ and (2.11), (2.12) hold simultaneously, we deduce in particular that every line is a line of Julia.

We shall prove that relatively weak conditions will suffice to yield the conclusions of Theorem 2, Corollary. We have

THEOREM 3. *If $f(z)$ is a transcendental integral function of finite (lower) order, then (2.11) and (2.12) hold simultaneously for all the functions $f_a(z)$ on a set E of (upper) logarithmic density 1, provided that $f(z)$ satisfies the Fabry gap condition (1.2).*

For functions of infinite order the same conclusions hold on a set E of logarithmic density 1 provided that

$$\varphi(n) = o\{n(\log n)^{-1}(\log\log n)^{-\alpha}\}, \quad \text{as } n \to \infty, \tag{2.15}$$

for some $\alpha > 2$.

We note that in all these cases every line is a line of Julia, and there are no exceptional values. Thus Pólya's theorem holds in this strengthened form for all functions of finite lower order, including zero order.

The fact that (2.11) holds under the hypotheses of Theorem 3 is due to Fuchs ([**5**]) for functions of finite order. The conclusion for functions of finite lower order represents a slight sharpening of a result of Sons ([**14**]), who proved a corresponding theorem on a set of infinite logarithmic measure. The condition (2.15) for (2.11) was suggested by Kövari ([**9**]), who carried out the proof under the slightly stronger condition

$$\varphi(n) = O(n)(\log n)^{-\alpha} \quad (\alpha > 2),$$

but obtained the result outside a set of finite logarithmic measure.

2.2. The rest of the paper will be devoted to the proof of Theorem 3 apart from the next section in which we prove Theorem 2. In §4 we obtain the lower bounds for $L(r,f)$ which are required for Theorem 3. The methods here are familiar and are due to Kövari ([**8**], [**9**]), Fuchs ([**5**]), and Sons ([**14**]). However, in order to prove Theorem 3 in the case of finite lower order we have to be rather careful to show that $L(r,f)$ is large at most points of certain particular intervals $[1, R]$, namely those for which $\log M(R,f)$ is not too big (Theorem 4) and this needs a modification in the arguments employed by the previous authors. The key for this is the use of Lemma 4 and in particular (4.5) subject to (4.4). For the application to infinite order (Theorem 5) we use (4.3). We note that the Wiman–Valiron method is not in fact essential for this type of application but can be replaced by the simpler Lemma 2, which gives an estimate for the remainder after N terms of a power series $f(z)$ on $|z| = r$ in terms of $M(R,f)$ for $R > r$ and, when combined with Lemma 4 permits a better control of the location of the exceptional intervals.

In §§5–8 we estimate $I_0(r)$ and these estimates constitute the central idea of the paper. In Lemma 7 we show that the argument of a polynomial is bounded on any ray by $k\pi$, where k is the number of distinct non-zero terms. This leads to a bound for the number of zeros of such a polynomial in a sector (Theorem 6). The bounds for the argument extend from polynomials $P(z)$ to power series $f(z) = P(z) + r(z)$, on a straight line segment $[0, Re^{i\theta}]$ provided that $|r(z)| < |f(z)|$ on the segment, and in particular if $|r(z)|$ is less than the minimum $L_1(\theta, R)$ of $|f(z)|$ on this segment (Lemma 8). Using a bound from a previous paper for the average of $-\log L_1(\theta, R)$ we finally obtain in Lemma 10 an estimate for the average value of $v(R, \theta)$. Unfortunately, we need the logarithmic integral of $v(r, \theta)$, and so the logarithmic integral of the estimate of Lemma 10. The result is achieved by means of Theorem 7, an inequality for real functions whose statement and proof fill §6. In §7 we apply Theorem 7 and Lemma 10 to obtain the required estimates for $I_0(r)$ for functions of infinite order, and in §8 for functions of finite lower order.

Finally, in § 9 the results of § 5 are combined with those in §§ 7 and 8 to complete the proof of Theorem 3.

3. Proof of Theorem 2

The proof of Theorem 2 is quite simple. We define ε_0 by the equation:

$$I_0(r) = \varepsilon_0^2\left(\frac{1}{\pi}-\frac{1}{4\pi^2}\right)\log M(r). \tag{3.1}$$

We put

$$J(\theta_1, \theta_2) = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\log|f(re^{i\theta})|\,d\theta,$$

and write $N(\theta_1, \theta_2)$ for $N(r, \theta_1, \theta_2, 0)$. We show first that

$$N(\theta_1, \theta_2) \geqslant J(\theta_1, \theta_2) - 2\varepsilon_0 \log M(r). \tag{3.2}$$

If $\varepsilon_0 > \frac{1}{2}(\theta_2 - \theta_1)$ this is trivial since then the right-hand side is negative. We now assume that $\varepsilon_0 \leqslant \frac{1}{2}(\theta_2 - \theta_1)$. With this assumption suppose that (3.2) is false. Then we have when $0 < \varepsilon < \varepsilon_0$, in view of (2.6),

$$\begin{aligned}
|V(r, \theta_1+\varepsilon)| + |V(r, \theta_2-\varepsilon)| &\geqslant |V(r, \theta_2-\varepsilon) - V(r, \theta_1+\varepsilon)| \\
&\geqslant J(\theta_1+\varepsilon, \theta_2-\varepsilon) - N(\theta_1+\varepsilon, \theta_2-\varepsilon) \\
&\geqslant J(\theta_1, \theta_2) - \frac{\varepsilon}{\pi}\log M(r) - N(\theta_1, \theta_2) \\
&> \left(2\varepsilon_0 - \frac{\varepsilon}{\pi}\right)\log M(r),
\end{aligned}$$

since (3.2) is false by hypothesis. We integrate this inequality with respect to ε from $\varepsilon = 0$ to ε_0, and deduce that

$$\varepsilon_0^2\left(2 - \frac{1}{2\pi}\right)\log M(r) < \int_0^{\varepsilon_0}|V(r, \theta_1+\varepsilon)|\,d\theta + \int_0^{\varepsilon_0}|V(r, \theta_2-\varepsilon)|\,d\theta \leqslant 2\pi I_0(r),$$

which contradicts (3.1). Thus (3.2) must hold.

We deduce at once that

$$N(\theta_2, \theta_1+2\pi) \geqslant J(\theta_2, \theta_1+2\pi) - 2\varepsilon_0 \log M(r).$$

Thus

$$\begin{aligned}
N(\theta_1, \theta_2) &= N(\theta_1, \theta_1+2\pi) - N(\theta_2, \theta_1+2\pi) \\
&= J(\theta_1, \theta_1+2\pi) - N(\theta_2, \theta_1+2\pi) \\
&\leqslant J(\theta_1, \theta_1+2\pi) - J(\theta_2, \theta_1+2\pi) + 2\varepsilon_0 \log M(r) \\
&= J(\theta_1, \theta_2) + 2\varepsilon_0 \log M(r).
\end{aligned}$$

On combining this with (3.2) we deduce that

$$|N(\theta_1, \theta_2) - J(\theta_1, \theta_2)| \leqslant 2\varepsilon_0 \log M(r) = \frac{4\pi}{\sqrt{(4\pi-1)}}(I_0(r) \log M(r))^{\frac{1}{2}}.$$

This yields (2.10), i.e. Theorem 2.

Next we have always

$$\frac{\theta_2-\theta_1}{2\pi}\log L(r) \leqslant \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \log|f(re^{i\theta})|\,d\theta \leqslant \frac{\theta_2-\theta_1}{2\pi}\log M(r),$$

so that (2.10), (2.11), and (2.12) yield (2.13). This proves the corollary.

4. Lower bounds for $L(r, f)$

We proceed to obtain suitable conditions for (2.11) to hold in order to prove Theorem 3. Following Kövari ([8], [**9**]), Fuchs ([**5**]), and Sons ([**14**]), we shall use for this an important inequality of Turan ([**15**]). This is

LEMMA 1. *Let* $P(z) = b_0 + b_1 z^{\lambda_1} + \ldots + b_{N-1} z^{\lambda_{N-1}}$ *be a polynomial of N terms, let* γ *be the arc* $z = re^{i\theta}$ $(\theta_1 < \theta < \theta_1 + \delta)$, *and let*

$$M_\gamma(r) = \max_{z \in \gamma}|P(z)|, \quad M(r) = \max_{|z|=r}|P(z)|.$$

Then

$$M(r) \leqslant \left(\frac{4e\pi}{\delta}\right)^N M_\gamma(r). \tag{4.1}$$

We apply the above lemma, by taking for $P(z)$ a suitable partial sum of our power series. A suitable error term for the remainder is provided by the following simple

LEMMA 2. *Suppose that* $f(z) = \sum_0^\infty a_n z^n$ *is regular in* $|z| < \rho_0$, *and that* $r < re^h < \rho_0$. *Then if*

$$f_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$$

we have, for $|z| = r$,

$$|f_N(z)| \leqslant \frac{M(re^h)e^{-Nh}}{h}.$$

In fact we have from Cauchy's inequality, applied with $R = re^h$,

$$|a_n| R^n \leqslant M(R),$$

so that

$$|a_n| r^n \leqslant M(R)\left(\frac{r}{R}\right)^n = M(re^h)e^{-nh}.$$

Thus

$$\sum_{n=N+1}^{\infty} |a_n| r^n \leqslant \frac{M(re^h)e^{-(N+1)h}}{(1-e^{-h})} = \frac{M(re^h)e^{-Nh}}{e^h-1} < \frac{M(re^h)e^{-Nh}}{h}.$$

We deduce

LEMMA 3. *Suppose that with the hypotheses of Lemma 2, the quantity N is chosen so large that*

$$N \geqslant \frac{1}{h}\left\{\log M(re^h) + \log^+\left(\frac{1}{h}\right)\right\}.$$

Then, if γ is the arc $z = re^{i\theta}$ ($\theta_1 < \theta < \theta_1+\delta$), and

$$M_\gamma(r,f) = \max_{z\in\gamma} |f(z)|,$$

we have

$$M_\gamma(r,f) \geqslant \left(\frac{\delta}{4\pi e}\right)^{\varphi(N)+1} M(r,f) - 2,$$

where $\varphi(N)$ is the number of non-zero terms among a_1 to a_N.

We apply Lemma 1 to $P_N(z) = f(z) - f_N(z)$. By Lemma 2 we have $|f_N(z)| \leqslant 1$ on $|z| = r$. Thus

$$M(r,f) \leqslant M(r,P_N)+1 \leqslant \left(\frac{4\pi e}{\delta}\right)^{\varphi(N)+1} M_\gamma(r,P_N)+1$$

$$\leqslant \left(\frac{4\pi e}{\delta}\right)^{\varphi(N)+1} \{M_\gamma(r,f)+1\}+1$$

$$\leqslant \left(\frac{4\pi e}{\delta}\right)^{\varphi(N)+1} \{M_\gamma(r,f)+2\},$$

and this proves Lemma 3.

4.1. *A lemma on growth.* We next need a lemma on growth for general increasing functions. The result is

LEMMA 4. *Suppose that $T(r)$ is a continuous increasing positive function of r for $r \geqslant 1$, and that $\varphi(x)$ is a continuous positive decreasing function of x for $x \geqslant 0$ such that,*

$$\int_1^\infty \frac{\varphi(x)\,dx}{x} < \infty. \tag{4.2}$$

Then, if $K > 0$, we have, for $r \geqslant 1$ and r outside a set of finite logarithmic measure,

$$T\{r[1+\varphi(T(r))]\} < e^K T(r). \tag{4.3}$$

If further λ *and* h *are positive constants, such that* $h < K/\lambda$, *and for some arbitrarily large* R *we have*

$$T(R) < R^{\lambda}, \tag{4.4}$$

then we have

$$T(re^{h}) < e^{K}T(r) \tag{4.5}$$

except on a subset E_R *of* $[1, R]$, *whose logarithmic measure is at most* $h\lambda K^{-1}\log R + O(1)$, *as* $R \to \infty$ *through values on which* (4.4) *holds.*

The result (4.3) subject to (4.2) is due to Edrei and Fuchs ([4], Lemma 10.1), but we include the proof for completeness.

Let r_1 be the smallest value of r for which (4.3) is false and, if r_n has already been defined, let r_{n+1} be the smallest value of r, if any, such that

$$r \geqslant r_n[1+\varphi\{T(r_n)\}]$$

and (4.3) is false. If there is no such r, for some n, then (4.3) holds for all sufficiently large r, and there is nothing more to prove. Thus we may assume that r_n exists for all n. Then we have, since $T(r)$ increases with r,

$$T(r_{n+1}) \geqslant T(r_n[1+\varphi\{T(r_n)\}]) \geqslant e^{K}T(r_n) \geqslant \ldots \geqslant e^{nK}T(r_1).$$

Thus $r_n \to \infty$ with n. We also see that (4.3) holds except in the intervals $[r_n, r'_n]$, where

$$r'_n = r_n[1+\varphi(T_n)], \quad T_n = T(r_n) \geqslant e^{K}T_{n-1}.$$

Also,

$$\int_{r_n}^{r'_n}\frac{dt}{t} = \log[1+\varphi(T_n)] \leqslant \varphi(T_n) \leqslant \frac{1}{K}\int_{T_{n-1}}^{T_n}\varphi(x)\frac{dx}{x},$$

since $\varphi(x)$ decreases with x. Thus the total logarithmic measure of the exceptional intervals $[r_n, r'_n]$ for $n \geqslant 2$ is at most

$$\frac{1}{K}\sum_{n=2}^{\infty}\int_{T_{n-1}}^{T_n}\varphi(x)\frac{dx}{x} = \frac{1}{K}\int_{T_1}^{\infty}\varphi(x)\frac{dx}{x} < \infty$$

by (4.2). Thus (4.3) holds outside a set of finite logarithmic measure.

Next we define a sequence r_n similarly with respect to (4.5), i.e. we choose for r_1 the smallest value of $r \geqslant 1$ for which (4.5) is false, and, if r_n has been defined, we choose for r_{n+1} the smallest value of $r \geqslant r_n e^{h}$, such that (4.5) is false.

Then again (4.5) holds except in the intervals $[r_n, r'_n]$, where

$$r'_n = r_n e^{h}, \quad T_n = T(r_n) \geqslant T(r'_{n-1}) \geqslant e^{K}T_{n-1}, \quad n \geqslant 2.$$

Let n be the largest integer for which $r_n \leqslant R$. Then

$$e^{(n-1)K}T_1 \leqslant T_n \leqslant T(R) < R^{\lambda},$$

so that

$$(n-1)K < \lambda \log R + \log \frac{1}{T_1},$$

$$nK < \lambda \log R + K + \log \frac{1}{T_1} = \lambda \log R + O(1).$$

Also (4.5) holds in $[1, R]$ outside the intervals $[r_\nu, r'_\nu]$ $(1 \leqslant \nu \leqslant n)$ whose total logarithmic measure is at most

$$nh = \frac{h}{K} nK < \frac{h\lambda}{K} \log R + O(1).$$

This completes the proof of Lemma 4.

4.2. *A lemma of Fuchs.* In order to achieve our lower bounds for $L(r,f)$ we follow the method of Fuchs and quote the following result of his ([**5**]).

LEMMA 5. *Suppose that* $0 < \eta < \frac{1}{2}$ *and* $0 < \delta < \frac{1}{2}$. *Let* $f(z)$ *be meromorphic in* $|z| \leqslant R$, *and such that* $f(0) = 1$, *and let* $d_1, d_2, \ldots, d_m$ *be the set of all zeros and poles of* $f(z)$ *in* $|z| \leqslant R$. *Then we have, for* $r < R$ *outside an exceptional set* F *of intervals of total length at most* $2\eta R$,

$$(4.6) \quad S = r \int_J \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta < \frac{4Rr}{(R-r)^2} \delta T(R,f) + \left(7 + \log \frac{\eta R}{\delta r}\right) \frac{2\delta m}{\eta} + \frac{2R}{r},$$

where J *is any interval of length* δ, *and* $T(r,f)$ *is the Nevanlinna characteristic.*

We apply this lemma to our integral functions $f(z)$, and shall replace $T(r,f)$ by the larger $T(r) = \log M(r,f)$. We suppose that $0 < h < 1$, and shall use Lemma 4. We suppose that (4.5) holds, i.e. that

$$(4.7) \qquad T(R_0 e^h) < e^K T(R_0),$$

assume that $\eta < \frac{1}{10}$, apply (4.6) with ηh instead of η, and put

$$R = R_0 e^{(1-\eta)h}, \quad R_1 = R_0 e^h, \quad R_2 = R_0 e^{(1-2\eta)h}.$$

Then if m is the number of zeros of $f(z)$ in $|z| \leqslant R$, we have

$$m \leqslant \frac{1}{\eta h} \int_R^{R_1} n(t) \frac{dt}{t} \leqslant \frac{T(R_1)}{\eta h}.$$

Also if $R_0 \leqslant r \leqslant R_2$, we have

$$\frac{4Rr}{(R-r)^2} \leqslant \frac{4R_1^2}{(R-R_2)^2} = \frac{4e^{2h}}{e^{2(1-2\eta)h}(e^{\eta h}-1)^2} < \frac{4e^{4\eta h}}{\eta^2 h^2} < \frac{8}{\eta^2 h^2}.$$

Thus we have in the interval $[R_0, R_2]$, outside a set of r of length at most $2\eta hR_1$,

$$S < \frac{8}{\eta^2h^2}\delta e^K T(R_0) + \left(7 + h + \log\frac{1}{\delta}\right)\frac{2\delta e^K T(R_0)}{\eta^2 h^2} + 2e.$$

Thus we have, for large R_0 and an absolute constant A,

$$S < Ae^K\delta \log\frac{1}{\delta}T(R_0)\frac{1}{\eta^2h^2},$$

for any arc of length δr on $|z| = r$, when r belongs to a subset E of the interval $[R_0, R_1]$ of length at least

$$\begin{aligned} R_2 - R_0 - 2\eta hR_1 &= R_1 - R_0 - 2\eta hR_1 - (R_1 - R_2) \\ &> R_1 - R_0 - 4\eta hR_1. \end{aligned}$$

Thus if E' is the complement of E in $[R_0, R_1]$, we have

$$\int_{E'}\frac{dr}{r} < \frac{1}{R_0}\int_{E'} dr < \frac{4\eta hR_1}{R_0} < 4e\eta h.$$

We write η instead of $4e\eta$ and embody our result in

LEMMA 6. *Suppose that* $0 < \eta < 1$, $0 < h < 1$, *and that* $T(r) = \log M(r,f)$ *satisfies* (4.7). *Then we have, for* $R_0 \leqslant r \leqslant R_0 e^h$, *except for a set* E' *of logarithmic measure at most* ηh,

$$r\int_\gamma \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta < \frac{A_1 e^K}{\eta^2h^2}\delta \log\frac{1}{\delta}T(R_0), \tag{4.8}$$

where A_1 *is an absolute constant, and* γ *is any arc on* $|z| = r$ *of length* δr, *where* $\delta < \frac{1}{2}$.

4.3. *Application to functions of finite order.* We now put

$$\Delta = \varlimsup_{n\to\infty}\frac{\varphi(n)}{n}, \tag{4.9}$$

where $\varphi(n)$ is the number of non-zero coefficients among $a_1, a_2, \ldots, a_n$, and have

THEOREM 4. *Suppose that* $f(z)$ *is an integral function such that for some arbitrarily large* R *we have*

$$\log M(R,f) < R^\lambda, \tag{4.10}$$

where λ *is a positive constant. Let* η_1, η_2 *be constants between* 0 *and* 1 *and suppose that*

$$\Delta \leqslant \frac{A_2\eta_1\eta_2}{(1+\lambda)(1+\log^+\lambda - \log\eta_1\eta_2)}, \tag{4.11}$$

where A_2 *is a certain absolute constant.*

Then there exists a subset E of the real axis, such that the logarithmic measure of $E[1,R]$ is at least $(1-\eta_1)\log R+O(1)$, as $R\to\infty$ through values satisfying (4.10) *and such that we have, for r in E,*

$$\log L(r,f) > (1-\eta_2)\log M(r,f). \tag{4.12}$$

A somewhat weaker inequality was proved by Anderson, but never published.† In order to achieve (4.12) he needed

$$\delta = \frac{O(1)}{\lambda^2\log\lambda}$$

for large λ, where δ was a larger upper density of the coefficients. We note that if

$$\Delta < A_2[(1+\lambda)(1+\log^+\lambda)]^{-1},$$

then $L(r,f)$ is unbounded by (4.11), (4.12), with η_1, η_2 near to one, so that $f(z)$ has no finite asymptotic or deficient value. For this case we can apply (4.11) with suitable $\eta_1 < 1$ and $\eta_2 < 1$.

We choose $K = 1$ in Lemma 4 and define h by the equation

$$h = \inf\left(1, \frac{\eta_1}{3\lambda}\right). \tag{4.13}$$

Then if (4.10) holds and $R > R_0$, (4.5) holds outside a set E_R in $[1,R]$, whose logarithmic measure is at most $\frac{1}{3}\eta_1\log R+O(1)$. We take for R_1 the lower bound of numbers $R > 1$, such that (4.5) holds, and if R_n has been defined, we define

$$R_n' = e^h R_n, \quad \text{so that } T(R_n') \leqslant eT(R_n), \tag{4.14}$$

and take for R_{n+1} the lower bound of all numbers $R \geqslant R_n'$, satisfying (4.5). Let E_1 be the set of complementary intervals $[R_n', R_{n+1}]$ and $E_1[1,R]$ the part of E_1 in $[1,R]$. Then $E_1[1,R]$ consists entirely of points where (4.5) is false and so by Lemma 4 we have

$$\int_{E_1(1,R)} \frac{dt}{t} \leqslant \frac{\eta_1}{3}\log R+O(1), \tag{4.15}$$

if R is such that (4.10) holds. Also the complement of $E_1(1,R]$ in $[1,R]$ consists of the intervals $[R_n, R_n']$ satisfying (4.14). In view of Lemma 6 we have, for r in $[R_n, R_n']$ and outside a subset of logarithmic measure $\frac{1}{3}\eta_1 h$,

$$r\int_\gamma \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta < \frac{A_3}{\eta_1{}^2h^2}\delta\log\frac{1}{\delta}T(R_n), \tag{4.16}$$

since we may take $\eta = \frac{1}{3}\eta_1$ in Lemma 6.

† The result is mentioned in a survey article by Fuchs ([6], Theorem 18, p. 286).

We next apply Lemma 4, again with h given by (4.13) and deduce that we have, outside a set E_2, such that (4.15) holds for $E_2(1,R)$ also,

$$T(re^h) < eT(r).$$

Thus outside E_2 we now apply Lemma 3, provided that

$$N \geqslant \frac{1}{h}\left(T(re^h) + \log^+\frac{1}{h}\right),$$

and in particular if r is large enough, $R_n < r < R_n'$, and

$$N \geqslant \frac{9}{h}T(R_n).$$

We also suppose given $\varepsilon > 0$, and suppose N so large that

$$\varphi(N)+1 < (\Delta+\varepsilon)N.$$

Then we have, for any arc γ of length $r\delta$ on $|z| = r$,

$$M_\gamma(r,f) \geqslant \left(\frac{\delta}{4\pi e}\right)^{(\Delta+\varepsilon)N} M(r,f) - 2,$$

so that for some point $z_0 = re^{i\theta_0}$ on γ we have

$$\log|f(z_0)| = \log M_\gamma(r,f) > T(r) - \frac{10(\Delta+\varepsilon)}{h}T(R_n)\log\left(\frac{4\pi e}{\delta}\right) \tag{4.17}$$

$$> T(r)\left(1 - \frac{10(\Delta+\varepsilon)}{h}\log\frac{4\pi e}{\delta}\right).$$

Hence if $z_1 = re^{i\theta_1}$ is any other point on γ, it follows that, if (4.16) holds,

$$\log|f(z_1)| \geqslant \log|f(z_0)| - r\left|\int_{\theta_0}^{\theta_1}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|d\theta\right|$$

$$\geqslant T(r)\left(1 - \frac{10(\Delta+\varepsilon)}{h}\log\frac{4\pi e}{\delta} - \frac{A_3}{\eta_1{}^2h^2}\delta\log\frac{1}{\delta}\right).$$

We recall that h is given by (4.13) and choose δ so small that

$$\frac{A_3\delta\log(1/\delta)}{\eta_1{}^2h^2} < \tfrac{1}{2}\eta_2, \tag{4.18}$$

which is certainly satisfied if

$$\delta < A_4\eta_2{}^2\eta_1{}^4h^4$$

and so if

$$\delta = A_5\eta_2{}^2\eta_1{}^8(1+\lambda^{-4}),$$

where A_5 is a suitable absolute constant. We then suppose that Δ is so small that

$$\frac{10\Delta}{h}\log\frac{4\pi e}{\delta} < \tfrac{1}{2}\eta_2,$$

and so

$$\frac{10(\Delta+\varepsilon)}{h}\log\frac{4\pi e}{\delta} < \tfrac{1}{2}\eta_2, \quad \text{for a suitable positive } \varepsilon.$$

This is certainly satisfied if

$$\Delta = \frac{A_6\eta_2 h}{\log(4\pi e/\delta)} < \frac{A_7\eta_1\eta_2}{(1+\lambda)(1+\log^+\lambda-\log\eta_1\eta_2)}.$$

Thus if (4.11) holds with a suitable A_2 the above conditions are satisfied. If, further, r satisfies (4.16), then, if $z_1 = re^{i\theta_1}$ is any point of $|z| = r$, we can always find θ_0, such that $\theta_0 < \theta_1 < \theta_0+\delta$ and such that $z_0 = re^{i\theta_0}$ satisfies (4.17), where δ satisfies (4.18) and we deduce that

$$\log|f(z_1)| > T(r)(1-\tfrac{1}{2}\eta_2-\tfrac{1}{2}\eta_2) = (1-\eta_2)T(r).$$

This is true for any point z_1 on $|z| = r$, so that

$$\log L(r,f) > (1-\eta_2)T(r) = (1-\eta_2)\log M(r,f)$$

as required.

Finally, the conclusion holds for all r not in E_1 nor E_2 but in $[R_n, R_n']$ with the exception of a set of logarithmic measure $\tfrac{1}{3}\eta_1 h$. Thus the logarithmic measure of the set of r in $[1, R]$ for which (4.12) is false is at most

$$O(1)+\tfrac{1}{3}\eta_1\sum_{\nu=1}^{n}\log\left(\frac{R_\nu'}{R_\nu}\right)+\int_{E_1(1,R)}\frac{dt}{t}+\int_{E_2(1,R)}\frac{dt}{t} < \eta_1\log R+O(1),$$

as $R\to\infty$. Here n is the least integer such that $R_n < R$. This proves Theorem 4.

4.4. *Application to functions of infinite order.* We proceed to prove

THEOREM 5. *Suppose that $f(z)$ is an integral function satisfying* (2.15). *Then we have*

$$\log L(r) \sim \log M(r)$$

as $r\to\infty$ on a set E of logarithmic density 1.

We again choose $T(r) = \log M(r)$, and apply Lemma 4 with

$$\varphi(x) = \frac{4}{\log x(\log\log x)^{1+\varepsilon}}, \quad x \geqslant 10, \quad x = \tfrac{3}{2}, \quad \varepsilon < \alpha-2.$$

We deduce that outside a set E_1 of finite logarithmic measure we have

$$T(re^h) < T\{r(1+2h)\} < \tfrac{3}{2}T(r),$$

where

$$h = h(r) = \frac{2}{\log T(r)\{\log\log T(r)\}^{1+\varepsilon}}.$$

We now apply Lemma 3 with this value of h, and deduce that, if

$$N > \frac{1}{h}\left\{T(re^h) + \log^+ \frac{1}{h}\right\},$$

i.e. certainly if r is large and

$$N = \tfrac{7}{8}T(r)\log T(r)(\log\log T(r))^{1+\varepsilon} + O(1),$$

we have, with the notation of Lemma 3, and on using (2.15)

$$\log M_\gamma(r,f) > T(r) - (\varphi(N)+1)\log(4\pi e/\delta) + o(1)$$

$$(4.19) \qquad > T(r)[1-(\log\log T(r))^{1+\varepsilon-\alpha}\log(4\pi e/\delta)],$$

where γ is any arc of length $r\delta$ on $|z| = r$.

Next we construct a set of intervals $[R_n, R_n']$ as before such that R_1 is the smallest number such that

$$T(R_1 e^h) \leqslant \tfrac{3}{2}T(R_1),$$

and if R_n has already been defined we set $R_n' = R_n e^h$, and define R_{n+1} to be the lower bound of numbers $R \geqslant R_n'$ such that

$$(4.20) \qquad T(Re^h) \leqslant \tfrac{3}{2}T(R).$$

Then if E_2 is the set of complementary intervals (R_n', R_{n+1}), we see that (4.20) is false in E_2, so that E_2 has finite logarithmic measure by Lemma 4. Also it follows from Lemma 6 that we have, for r in (R_n, R_n') except in a subset E_n of logarithmic measure ηh,

$$r\int_\gamma \left|\frac{f'(re^{i\theta})}{f(r^{i\theta})}\right| d\theta < \frac{2A_1}{\eta^2 h^2}\delta \log\frac{1}{\delta} T(R_n).$$

We choose

$$\eta = \frac{1}{\log T(R_n)}, \quad \delta = [\log T(R_n)]^{-6},$$

and deduce that if γ is our arc of length $r\delta$, where r is in $[R_n, R_{n+1}]$ and outside E_n, then we have for large n

$$r\int_\gamma \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta < A\{\log T(R_n)\}^4\{\log\log T(R_n)\}^{3+2\varepsilon}(\log T(r_n))^{-6}T(r_n)$$

$$< \eta T(R_n).$$

Suppose now that r satisfies this condition and let $z_0 = re^{i\theta_0}$ be any point on $|z| = r$. Then by (4.19) we can find $z_1 = re^{i\theta_1}$ on $|z| = r$ such that $\theta_0 < \theta_1 \leqslant \theta_1 + \delta$ and

$$\log|f(z_1)| > T(r)\{1 - 6[\log\log T(R_n) + O(1)](\log\log T(R_n))^{1+\varepsilon-\alpha}\}$$

$$> T(r)[1 - 7\log\log T(R_n)^{2+\varepsilon-\alpha}].$$

Hence

$$\log|f(z_0)| > T(r)[1-\eta-7\log\log T(R_n)^{2+\varepsilon-\alpha}].$$

We may suppose that $|f(z_0)| = L(r)$ and deduce that

$$\log L(r) \sim T(r), \quad \text{as } r \to \infty,$$

through such values. Also the set of exceptional values of r consists of a set E of finite logarithmic measure, together with all the sets E_n, which occupy a proportion of $[R_n, R_n']$ which is small for large n. Thus the exceptional set has logarithmic density zero, as required. This completes the proof of Theorem 5.

5. Estimates for $I_0(r)$: the basic formalism

Having obtained our estimates for $L(r)$ we now turn to $I_0(r)$.

The basis of our results is the following lemma for polynomials.

LEMMA 7. *Suppose that $f(z) = 1+\sum_{\nu=1}^{N} a_\nu z^\nu$ is a polynomial and that k of the terms a_ν are different from zero. Then if $f(z) \neq 0$ on the ray $\arg z = \theta$, we have*

$$|v(\rho,\theta)| \leqslant k\pi \quad (0<\rho<\infty).$$

We assume, as we may do without loss of generality, that $\theta = 0$, since otherwise we may consider $f(ze^{i\theta})$ instead of $f(z)$. We write $a_\nu = b_\nu + ic_\nu$, so that

$$f(z) = 1+\sum_{\nu=1}^{N} b_\nu z^\nu + i\sum_{\nu=1}^{N} c_\nu z^\nu \tag{5.1}$$

$$= P(z)+iQ(z),$$

say, where $P(z)$, $Q(z)$ are real for real z. By Descartes's rule of signs† $Q(x)$ has at most $k-1$ positive zeros, since $Q(x)$ is a real polynomial having at most k distinct terms. These zeros divide the positive real axis into at most k intervals in each of which $Q(x)$ has constant sign, and so the argument of $P(x)+iQ(x)$ varies by at most π in each of these intervals. This proves Lemma 7.

We can deduce immediately the following result, which may have independent interest.‡

THEOREM 6. *Let $P(z)$ be a polynomial of degree N, having $k+1$ non-zero terms and such that $P(0) \neq 0$. Let $n(\rho,\theta_1,\theta_2)$ be the number of zeros of $P(z)$ in the sector $S(\rho,\theta_1,\theta_2)$. Then, given $\varepsilon > 0$, we have, for sufficiently large*

† See, for instance, Pólya and Szegö ([**13**], Problem 36, p. 43).

‡ Biernacki implies in a footnote to [**3**] that something similar to Theorem 6 would follow from the method used by him in [**2**], pp. 587–94. However, his arguments in [**2**] are not easy to follow.

ρ, and $\theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$,

$$\left| n(\rho, \theta_1, \theta_2) - N\frac{\theta_2 - \theta_1}{2\pi} \right| < k + \varepsilon. \tag{5.2}$$

In particular $P(z)$ has at least one zero in the closure of $S(\rho, \theta_1, \theta_2)$ for large ρ, provided that

$$\frac{\theta_2 - \theta_1}{2\pi} \geqslant \frac{k}{N}. \tag{5.3}$$

We consider the variation v_0 of $\arg P(z)$ around the boundary of $S(\rho, \theta_1, \theta_2)$. By Lemma 7 the arms $\arg z = \theta_1, \theta_2$, contribute at most $2k\pi$ to v_0. Also on $|z| = \rho$ for large ρ

$$P(z) = z^N e^{iN\theta}(a_N + O(1)/\rho),$$

so that the arc on $|z| = \rho$ contributes $N(\theta_2 - \theta_1) + O(1)/\rho$ to v_0. Thus

$$\begin{aligned} |v_0 - N(\theta_2 - \theta_1)| &= |2\pi n(\rho, \theta_1, \theta_2) - N(\theta_2 - \theta_1)| \\ &< 2k\pi + O(1)/\rho. \end{aligned}$$

This proves the result, provided that $P(z)$ has no zeros on the rays $\arg z = \theta_1, \theta_2$. If there are such zeros we replace θ_1, θ_2 by $\theta_1 + \delta$, $\theta_2 - \delta$, where δ is chosen so small that $N\delta/\pi < \varepsilon$, and $P(z) \neq 0$ for $\theta_1 < \arg z < \theta_1 + \delta$, and $\theta_2 - \delta < \arg z < \theta_2$. Then we may apply (5.2) with $\theta_1 + \delta$, $\theta_2 - \delta$ instead of θ_1, θ_2 and deduce (5.2) with 2ε instead of ε. This proves Theorem 6.

In particular (5.2) gives a contradiction if $n(\rho, \theta_1, \theta_2) = 0$ for all ρ, and

$$\frac{N(\theta_2 - \theta_1)}{2\pi} > k.$$

If $N(\theta_2 - \theta_1)/(2\pi) = k$, and $P(z)$ has no zeros for $\theta_1 \leqslant \arg z \leqslant \theta_2$, then $P(z)$ has no zeros for $\theta_1 - \delta < \arg z < \theta_2 + \delta$, for some positive δ, and so we again have a contradiction. This proves (5.3).

The inequality (5.3) is sharp, at least when N is a multiple of k. Suppose in fact that $N = pk$, where p is a positive integer, let $Q(z)$ be a polynomial of degree k, with all its zeros real and positive, e.g. $Q(z) = (z-1)^k$, and put

$$P(z) = Q(z^p).$$

Then $P(z)$ has $k+1$ non-zero terms and degree N and all the zeros of $P(z)$ lie on the rays $\arg z = 2\nu\pi/p$ $(0 \leqslant \nu \leqslant p-1)$. Thus $P(z)$ has no zero in $S(\rho, 0, 2\pi/p)$, so that (5.3) is false if we replace the closure of $S(\rho, \theta_1, \theta_2)$ by $S(\rho, \theta_1, \theta_2)$.

For our application we need a slight extension of Lemma 7. This is

LEMMA 8. *Suppose that the power series $f(z)$ given by* (1.1) *is regular in* $|z| \leqslant R$, and *non-zero on the segment* $\gamma(\theta)$: $z = te^{i\theta}$ $(0 < t < R)$.

Suppose further that

$$\left|\sum_{n+1}^{\infty} a_\nu z^\nu\right| < |f(z)|$$

on $\gamma(\theta)$. *Then* $|v(R,\theta)| \leqslant \pi(\varphi(n)+\frac{1}{2})$.

We write

$$P(z) = 1+\sum_{1}^{n} a_\nu z^\nu, \quad Q(z) = \sum_{n+1}^{\infty} a_\nu z^\nu,$$

so that $P(z)$ has $k+1 = \varphi(n)+1$ non-zero terms. Also on $\gamma(\theta)$ we put

$$\frac{P(z)}{f(z)} = \left(1-\frac{Q(z)}{f(z)}\right).$$

By hypothesis $|Q(z)/f(z)| < 1$, so that the argument of $1-Q(z)/f(z)$ is continuous on $\gamma(\theta)$ and less than $\frac{1}{2}\pi$ in absolute value. Thus

$$|\arg P(z) - \arg f(z)| < \tfrac{1}{2}\pi$$

on $\gamma(\theta)$. Now Lemma 8 follows from Lemma 7.

5.1. *An estimate for* $v(\rho,\theta)$. In order to apply Lemma 8 we need an estimate for the minimum $L_1(\theta,R)$ of $|f(z)|$ on the segment $\gamma(\theta,R)$. The following result was obtained by the author in a previous paper ([7], Theorem 1, p. 183].

LEMMA 9. *Suppose that $f(z)$ is meromorphic in $|z| \leqslant Re^h$, where $h > 0$, and that $f(0) = 1$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{L_1(\theta,R)}\,d\theta < (1+\psi(e^{-h}))T(Re^h,f),$$

where $T(R,f)$ denotes the Nevanlinna characteristic and

$$\psi(t) = \frac{(1-t)\log\{1+(2\pi\sqrt{t}/(1-t))\}}{\pi\sqrt{t}\log(1/t)}.$$

The result in Lemma 9 was in fact proved with $T(Re^h, 1/f)$ instead of $T(Re^h,f)$, but since $f(0) = 1$, these two quantities are the same. Also, since f is regular in our case, we have

$$T(Re^h,f) \leqslant \log M(re^h,f).$$

Next we note that, for $h \leqslant 1$,

$$\psi(e^{-h}) = \frac{2\sinh(\frac{1}{2}h)}{\pi h}\log\left(1+\frac{\pi}{\sinh\frac{1}{2}h}\right) < \frac{2\sinh\frac{1}{2}}{\pi}\log\left(1+\frac{2\pi}{h}\right)$$

$$< \frac{2\sinh\frac{1}{2}}{\pi}\left(\log\frac{1}{h}+\log(1+2\pi)\right) < \left(1+\tfrac{1}{2}\log\frac{1}{h}\right).$$

Thus we deduce from Lemma 9 that, for $h \leqslant 1$,

$$(5.4)\qquad \frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{L_1(\theta,R)}\,d\theta < \left(2+\tfrac{1}{2}\log^+\frac{1}{h}\right)\log M(Re^h,f).$$

The inequality evidently remains valid for $h > 1$, since in this case we may replace h by 1, which diminishes the right-hand side of (5.4). We deduce

LEMMA 10. *Suppose that $h > 0$, and that, for some ε such that $0 < \varepsilon \leqslant 1$, we have*

$$(5.5)\qquad \varphi(n) \leqslant \varepsilon n, \quad \text{for } n \geqslant \frac{\log M(Re^h)}{h}.$$

Then

$$(5.6)\qquad \frac{1}{2\pi}\int_0^{2\pi}|v(R,\theta)|\,d\theta \leqslant 5+\frac{\varepsilon}{h}\left(10+2\log^+\frac{1}{h}\right)(\log M(Re^h)+2).$$

We recall from Lemma 2 that

$$\sum_{\nu=n+1}^{\infty}|a_\nu|R^\nu \leqslant \frac{M(Re^h)e^{-nh}}{h}.$$

We choose for n the smallest positive integer such that

$$(5.7)\qquad \frac{M(Re^h)e^{-nh}}{\inf(1,h)} < L_1(\theta,R).$$

Then (5.5) holds, since $L_1(\theta,R) \leqslant f(0) = 1$. Thus Lemma 8 gives

$$|v(R,\theta)| \leqslant \pi(\varphi(n)+\tfrac{1}{2}) \leqslant \pi(\varepsilon n+\tfrac{1}{2}).$$

Also

$$e^{(n-1)h} \leqslant \frac{M(Re^h)}{L_1(\theta,R)\inf(1,h)},$$

so that

$$n \leqslant 1+\frac{1}{h}\left\{\log M(Re^h)+\log\frac{1}{L_1(\theta,R)}+\log^+\frac{1}{h}\right\}.$$

Thus we deduce that

$$(5.8)\qquad |v(R,\theta)| \leqslant \pi\left\{\frac{3}{2}+\frac{\varepsilon}{h}\left(\log M(Re^h)+\log\frac{1}{L_1(\theta,R)}+\log^+\frac{1}{h}\right)\right\}.$$

We now integrate with respect to θ and use (5.4). This gives

$$\frac{1}{2\pi}\int_0^{2\pi}|v(R,\theta)|\,d\theta \leqslant \pi\left\{\frac{3}{2}+\frac{\varepsilon}{h}\left[\left(3+\tfrac{1}{2}\log^+\frac{1}{h}\right)\log M(Re^h)+\log^+\frac{1}{h}\right]\right\}$$

$$< 5+\frac{\varepsilon}{h}\left(10+2\log^+\frac{1}{h}\right)(\log M(Re^h)+2).$$

This proves Lemma 10.

6. A theorem for real functions

In order to exploit Lemma 10, and thus to prove our results we need to integrate the estimate of (5.6) and show that the result is in general comparable with $\log M(R)$. The argument here depends solely on the convexity of the function $\log M(R)$ as a function of $\log R$. The result may have other applications and so we state it as

THEOREM 7. *Suppose that $g(x)$ is a positive strictly increasing convex function of x for $x > 0$, such that $g'(x) \to \infty$ with x. Let x_ν be defined by*

$$g(x_\nu) = 2^\nu g(0) \quad (\nu = 0, 1, \ldots), \tag{6.1}$$

put $h_\nu = x_{\nu+1} - x_\nu$ and $h(x) = x_{\nu+2} - x$, when $x_\nu \leqslant x < x_{\nu+1}$. We suppose that A, B are non-negative constants, and define

$$g_0(x) = g(x+h)/h,$$

$$g_1(x) = \left(A + B \log^+ \frac{1}{h}\right) g_0(x).(1 + (\log^+ g_0(x))^\alpha [\log^+ \log^+ g_0(x)]^\beta)^{-1}, \tag{6.2}$$

where $h = h(x)$, and α, β are real constants, such that $\alpha > 0$, or $\beta > 0$ and $\alpha = 0$. Let

$$G_1(x) = \int_0^x g_1(t)\, dt. \tag{6.3}$$

Then if $0 < \eta < 1$, there exists a positive constant K, depending only on α, β, and a set E having lower density at least $1 - \eta$, and depending only on η and the function $g(x)$, but not on α, β, A, B, such that we have, for $x \in E$,

$$G_1(x) < K \left(\log \frac{e}{\eta}\right)^{\alpha + |\beta| + 1} g(x) (\log^+ g_0(x)^{-\alpha}) \tag{6.4}$$

$$\times (\log^+ \log^+ g_0(x))^{-\beta} \left\{A + B\left(1 + \log^+ \frac{1}{h(x)}\right)\right\}.$$

We take for E' the union of all the intervals $[x_{\nu+1} - \frac{1}{2}\eta h_\nu, x_{\nu+1} + \frac{1}{2}\eta h_\nu]$, and for E the complement of E'. Suppose that $X > 0$ and that $x_n \leqslant X < x_{n+1}$. Then the part of E' in $[0, X]$ has measure at most

$$\eta \sum_{\nu=0}^{n-1} (x_{\nu+1} - x_\nu) + \tfrac{1}{2}\eta(X - x_n) \leqslant \eta X.$$

Thus E' has upper density at most η and so E has lower density at least $1 - \eta$.

We now assume that $x_n < x < x_{n+1}$ and that $x \in E$. Then we have, for $\nu < n$,

$$\eta 2^{\nu - n - 1} < \frac{h_{\nu+1}}{h_\nu} \leqslant 2. \tag{6.5}$$

To prove the right-hand inequality in (6.5), which is true for all ν, we note that since $g'(x)$ is increasing

$$\frac{g(x_{\nu+1})-g(x_\nu)}{h_\nu} \geqslant g'(x_\nu) \geqslant \frac{g(x_\nu)-g(x_{\nu-1})}{h_{\nu-1}}.$$

Thus

$$\frac{2^\nu g(0)}{h_\nu} \geqslant \frac{2^{\nu-1}g(0)}{h_{\nu-1}},$$

which gives the required result. We suppose now that, for some $\nu \leqslant n$,

$$\frac{h_{\nu+1}}{h_\nu} = \delta,$$

and proceed to obtain a lower bound for δ. Then we have, for $0 \leqslant k \leqslant n-\nu-1$,

$$h_{\nu+k+1} \leqslant 2h_{\nu+k} \leqslant 2^k h_{\nu+1} = 2^k \delta h_\nu.$$

Thus

$$x - x_{\nu+1} \leqslant x_{n+1} - x_{\nu+1} = \sum_{k=0}^{n-\nu-1} h_{\nu+k+1} \leqslant \delta h_\nu \sum_{k=0}^{n-\nu-1} 2^k$$
$$< 2^{n-\nu}\delta h_\nu.$$

But since $x \in E$, we have

$$x - x_{\nu+1} > \tfrac{1}{2}\eta h_\nu,$$

so that

$$\tfrac{1}{2}\eta < 2^{n-\nu}\delta,$$

so that

$$\delta > \eta 2^{\nu-n-1}.$$

This completes the proof of (6.5).

We note next that, for $p > 0$,

$$\frac{h_{n-p}}{h_n} = \prod_{k=0}^{p-1} \frac{h_{n-k-1}}{h_{n-k}},$$

so that (6.5) shows that

$$2^{-p} \leqslant \frac{h_{n-p}}{h_n} \leqslant \frac{2^{p(p+1)}}{\eta^p}. \tag{6.6}$$

Using (6.5) and (6.6) we shall proceed to estimate $G_1(x)$. It is sufficient to consider the cases $A = 1$, $B = 0$, and $A = 0$, $B = 1$, separately, since both sides of (6.4) are clearly linear in A, B. We accordingly define, for $A = 1$, $B = 0$,

$$I_\nu = \int_{x_\nu}^{x_{\nu+1}} g_1(t)\,dt \quad (0 \leqslant \nu < n),$$

$$I_n = \int_{x_n}^{x} g_1(t)\,dt;$$

and, for $A = 0$, $B = 1$,

$$I'_\nu = \int_{x_\nu}^{x_{\nu+1}} g_1(t)\,dt \quad (0 \leqslant \nu < n),$$

$$I'_n = \int_{x_n}^{x} g_1(t)\,dt.$$

Thus in the general case we have

$$G_1(x) = A\sum_{\nu=0}^{n} I_\nu + B\sum_{\nu=0}^{n} I'_\nu. \tag{6.7}$$

We denote by K any positive constant depending on α, β, only, not necessarily the same each time.

Suppose that $x_\nu < t < x_{\nu+1}$, where $\nu \leqslant n$, and $t \leqslant x$ if $\nu = n$. Then

$$g_0(t) = \frac{g(x_{\nu+2})}{x_{\nu+2}-t} = \frac{g(0)2^{\nu+2}}{h_{\nu+1}+x_{\nu+1}-t}. \tag{6.8}$$

Thus we have, using (6.5),

$$g_0(t) \geqslant \frac{g(0)2^{\nu+2}}{h_\nu + h_{\nu+1}} \geqslant \frac{g(0)2^{\nu+2}}{3h_\nu} = \frac{2^{\nu-n}g(x_{n+2})}{3h_\nu}.$$

On the other hand, since $x \in E$,

$$h(x) = h_{n+1} + x_{n+1} - x > \tfrac{1}{2}\eta h_n,$$

and by (6.6) we have, for $\nu = n-p$,

$$h_\nu < \frac{2^{p(p+1)}}{\eta^p} h_n.$$

Thus

$$g_0(t) > \frac{\eta^{p+1}2^{-p(p+1)-p-1}}{3}\frac{g(x_{n+2})}{h(x)} > \frac{1}{3}\left(\frac{\eta}{2}\right)^{(p+1)^2} g_0(x).$$

Hence

$$\log^+ g_0(x) < \log^+ g_0(t) + (p+1)^2 \log\frac{2}{\eta} + \log 3$$

$$< \left((p+1)^2 \log\frac{6}{\eta}\right)(1+\log^+ g_0(t)).$$

Thus if $\alpha > 0$, or $\alpha = 0$, $\beta > 0$,

$$(\log^+ g_0(x))^\alpha (\log^+\log^+ g_0(x))^\beta$$

$$< K\left[(p+1)^2\log\frac{6}{\eta}\right]^{\alpha+|\beta|}\{1+(\log^+ g_0(t))^\alpha(\log^+\log^+ g_0(t))^\beta\},$$

so that

$$(6.9)\quad \{1+(\log^+ g_0(t))^\alpha(\log^+\log^+ g_0(t))^\beta\}^{-1}$$
$$< K(p+1)^{2\alpha+2|\beta|}\left(\log\frac{6}{\eta}\right)^{\alpha+|\beta|}(\log^+ g_0(x))^{-\alpha}(\log^+\log^+ g_0(x))^{-\beta}.$$

We next obtain an upper bound for the other terms in the definition of $g_1(t)$. We have from (6.8), for $\nu = n-p < n$,

$$\int_{x_\nu}^{x_{\nu+1}} g_0(t)\,dt = g(0)2^{\nu+2}\int_{h_{\nu+1}}^{h_{\nu+1}+h_\nu}\frac{d\tau}{\tau}$$
$$= g(0)2^{\nu+2}\log\left(1+\frac{h_\nu}{h_{\nu+1}}\right)$$
$$= g(0)2^{n+2-p}\log\left(1+\frac{h_{n-p}}{h_{n-p+1}}\right)$$
$$< 4g(x)2^{-p}\log\left(1+\frac{2^{p+1}}{\eta}\right) < 4g(x)2^{-p}(p+1)\log\frac{3}{\eta}.$$

Also

$$\int_{x_n}^{x} g_0(t)\,dt = g(0)2^{n+2}\int_{h_{n+1}+x_{n+1}-x}^{h_{n+1}+h_n}\frac{d\tau}{\tau}$$
$$< g(0)2^{n+2}\log\frac{h_n}{x_{n+1}-x} < 4g(x)\log\frac{2}{\eta},$$

since $x \in E$, so that $x_{n+1}-x > \frac{1}{2}\eta h_n$.

Using (6.9), we deduce that, for $p = n-\nu$ $(0 \leqslant \nu \leqslant n)$,

$$(6.10)\quad I_\nu < Kg(x)(\log g_0(x))^{-\alpha}(\log^+\log^+ g_0(x))^{-\beta}$$
$$\times (p+1)^{2\alpha+2|\beta|+1}2^{-p}\left(\log\frac{6}{\eta}\right)^{\alpha+|\beta|+1}.$$

We next tackle I'_ν. We note that for $\nu = n-p$, where $p > 0$,

$$\int_{x_\nu}^{x_{\nu+1}} g_0(t)\log^+\frac{1}{h(t)}\,dt < \log^+\frac{1}{h_{\nu+1}}\int_{x_\nu}^{x_{\nu+1}} g_0(t)\,dt$$
$$< 4g(x)2^{-p}(p+1)\log\frac{3}{\eta}\left(\log^+\frac{1}{h_n}+(p-1)\log 2\right),$$

in view of (6.6). Since also $h(x) < h_n+h_{n+1} < 3h_n$ by (6.5), we deduce that

$$\int_{x_\nu}^{x_{\nu+1}} g_0(t)\log^+\frac{1}{h(t)}\,dt \leqslant Kg(x)2^{-p}(p+1)^2\log\frac{3}{\eta}\left(1+\log^+\frac{1}{h(x)}\right).$$

Similarly

$$\int_{x_n}^{x} g_0(t)\log^+\frac{1}{h(t)}\,dt \leqslant \log^+\frac{1}{h(x)}\int_{x_n}^{x} g_0(t)\,dt$$

$$< 4g(x)\left(\log^+\frac{1}{h(x)}\right)\log\frac{2}{\eta}.$$

These inequalities together with (6.9) yield, for $\nu = 0$ to n,

$$(6.11)\qquad I'_\nu < Kg(x)(\log^+ g_0(x))^{-\alpha}(\log^+\log^+ g_0(x))^{-\beta}$$

$$\times (p+1)^{2\alpha+2|\beta|+2}2^{-p}\left(\log\frac{6}{\eta}\right)^{\alpha+|\beta|+1}\left(1+\log^+\frac{1}{h(x)}\right).$$

Summing (6.10) and (6.11) from $p = 0$ to n and using (6.7), we deduce (6.4), with $6/\eta$ instead of e/η. We deduce the result with e/η by increasing K if necessary. This proves Theorem 7.

7. Estimates for $I_0(r)$ conclusion

We proceed to apply Theorem 7 to Lemma 10. We write, for any complex number a, $I_a(r)$ for the quantity $I_0(r)$, defined as in (2.9) but with respect to $f_a(z)$ instead of $f(z)$, where $f_a(z)$ is given by (2.14). We then have the following

THEOREM 8. *Suppose that $f(z)$ is the integral function* (1.1) *and let $\varphi(n)$ be the number of non-zero coefficients among a_1 to a_n. Suppose further that we have, for all sufficiently large n,*

$$(7.1)\qquad \varphi(n) < \varepsilon n(\log n)^{-\alpha}(\log\log n)^{-\beta},$$

where $\varepsilon > 0$, and $\alpha > 0$, with β real, or $\alpha = 0$, with $\beta > 0$. Let δ, η be numbers such that $0 < \delta < 1$, $0 < \eta < 1$, and a be a complex number such that $a = 1$, or $\delta < |a-1| < \delta^{-1}$.

Then there exists a set E of numbers on the positive axis, which depends only on $f(z)$ and has lower logarithmic density at least $1-\eta$, a constant K depending on α, β only, and r_0 depending on $\alpha, \beta, \delta, \varepsilon$, and $f(z)$ only, such that, if $r \in E$, and $r > r_0$, we have

$$(7.2)\qquad I_a(r) < K\varepsilon\left(\log\frac{e}{\eta}\right)^{\alpha+|\beta|+1}\left(\log^+\frac{1}{h(r)}+1\right)$$

$$\times \log M(r)\log N(r)^{-\alpha}(\log\log N(r))^{-\beta}.$$

Here $N(r)$ is the central index of $f(z)$ and $h = h(r)$ is a quantity such that

$$M(r)^2 < M(re^h) < M(r)^4, \tag{7.3}$$

where $M(r)$ is the maximum modulus of $f(z)$.

We start by applying Lemma 10 to $f_a(z)$. Suppose first that $a \neq 1$, and write

$$M_a(r) = M(r, f_a(z)).$$

Then since

$$f_a(z) = \frac{f(z)-a}{1-a}, \quad f(z) = a+(1-a)f_a(z),$$

we deduce that, if $\delta < |1-a| < \delta^{-1}$,

$$M_a(r) \leqslant \frac{M(r)+|a|}{|1-a|} \leqslant M(r)\frac{1+|a|}{|1-a|} \leqslant \frac{3}{\delta}M(r),$$

$$M(r) \leqslant |a|+\delta^{-1}M_a(r) \leqslant \frac{3}{\delta}M_a(r).$$

Suppose now that r_0 is so chosen that $M(r_0) > (3/\delta)^2$. Then we have

$$3 < M(r)^{\frac{1}{2}} < M_a(r) < M(r)^2 \quad (r > r_0). \tag{7.4}$$

We now take

$$g(x) = \log M(e^x) \tag{7.5}$$

in Theorem 7, let α, β be the quantities occurring in (7.1), and define $h = h(x)$ as in Theorem 7. It follows from these definitions that

$$2g(x) \leqslant g(x+h) \leqslant 4g(x),$$

so that (7.3) holds.

Next we write $x = \log r$,

$$g_0(x) = \frac{g(x+h)}{h}, \tag{7.6}$$

and deduce from (7.1) that, for $n > \frac{1}{2}g_0(x)$ $(x > x_1)$, we have

$$\begin{aligned} \varphi(n) &< \varepsilon n(\log n)^{-\alpha}(\log\log n)^{-\beta} \\ &< K\varepsilon n\{1 + (\log^+ g_0(x))^{\alpha}(\log^+\log^+ g_0(x)^{\beta}\}^{-1}. \end{aligned} \tag{7.7}$$

Also if we temporarily define $\log M_a(e^x) = g_a(x)$, it follows from (7.4) that, for $x > x_2$,

$$g_a(x) > \tfrac{1}{2}g(x), \quad \frac{g_a(x+h)}{h} > \frac{1}{2}\frac{g(x+h)}{h} = \tfrac{1}{2}g_0(x).$$

Thus we have (7.7), with the same values of h, provided that $x > x_2(\delta)$ and $n > g_a(x+h)/h$. Hence we deduce from Lemma 10, that, if

$R = e^x > R_0 = e^{x_0}$, say,

$$\frac{1}{2\pi}\int_0^{2\pi} |v_a(R,\theta)|\,d\theta$$
$$< 5 + K\varepsilon(1 + (\log^+ g_0(x))^\alpha (\log^+\log^+ g_0(x))^\beta)^{-1}$$
$$\times \frac{1}{h}\left(10 + 2\log^+\frac{1}{h}\right)(g_a(x+h)+2)$$
$$< 5 + K\varepsilon\left(1+\log^+\frac{1}{h}\right) g_0(x)\{1 + (\log^+ g_0(x))^\alpha (\log^+\log^+ g_0(x))^\beta\}^{-1}, \tag{7.8}$$

where $v_a(R,\theta)$ refers to the function $f_a(z)$.

We can also deduce (7.8) for $a = 1$. In this case

$$f_1(z) = \frac{f(z)-1}{cz^p}$$

so that (7.4) again holds for sufficiently large r, since $f(z)$ is a transcendental integral function. Thus the argument proceeds as before and we deduce (7.7) for $x > x_0$, provided that a, δ are related as in Theorem 8, and $x > x_2$, where x_2 depends on δ only. We note next that, since $g'(x)$ is increasing,

$$g(x+h) > g(x) + hg'(x),$$

so that

$$g_0(x) = \frac{g(x+h)}{h} > g'(x).$$

Also $g'(x)$ cannot tend to a finite limit, since otherwise $g(x) = O(x)$, so that $f(z)$ is a polynomial. Thus $g_0(x) \to \infty$ with x, and hence so do

$$g_0(x)\{1 + (\log^+ g_0(x))^\alpha (\log^+\log^+ g_0(x))^\beta\}^{-1} \quad \text{and} \quad g_1(x),$$

where $g_1(x)$ is defined as in (6.2).

It follows that we have, for $x > x_3$, from (7.8)

$$\frac{1}{2\pi}\int_0^{2\pi} |v_a(R,\theta)|\,d\theta < K\varepsilon\left(1+\log^+\frac{1}{h}\right) g_0(x)\{1 + (\log^+ g_0(x))^\alpha (\log^+\log^+ g_0(x))^\beta\}^{-1}.$$

Here x_3 depends on $K\varepsilon$ as well as $g_0(x)$, i.e. on α, β, δ, ε, and f.

We are now able to apply Theorem 7. We have

$$\varphi(x) = I_a(e^x) = I_a(e^{x_3}) + \int_{e^{x_3}}^{e^x} \frac{dt}{t}\frac{1}{2\pi}\int_0^{2\pi} |v_a(t,\theta)|\,d\theta$$
$$= I_a(e^{x_3}) + \int_{x_3}^{x} d\tau \frac{1}{2\pi}\int_0^{2\pi} |v_a(e^\tau,\theta)|\,d\theta$$
$$\leqslant I_a(e^{x_3}) + \int_0^x g_1(t)\,dt,$$

where $g_1(t)$ is as in (6.2) with $A = B = K\varepsilon$. We also note that, since $g_1(t) \to \infty$ with t,

$$\int_0^x g_1(t)\,dt \to \infty, \quad \text{as } x \to \infty. \tag{7.9}$$

We now deduce from Theorem 7, that we have, on the set E of that theorem, from (6.4)

$$I_a(e^x) < \dot{K}\varepsilon\left(\log\frac{e}{\eta}\right)^{\alpha+|\beta|+1} g(x)(\log g_0(x))^{-\alpha} \tag{7.10}$$

$$.(\log\log g_0(x))^{-\beta}\left(1+\log^+\frac{1}{h(x)}\right) + I_a(e^{x_3}).$$

It follows from (6.4) and (7.9) that the right-hand side tends to infinity with x. The set of $x = \log r$ has lower density at least $1-\eta$ and so the corresponding set of r has lower logarithmic density at least $1-\eta$.

It is not difficult to see from continuity considerations, that $I_a(e^{x_3})$ is uniformly bounded, when x_3 is fixed and a varies subject to $\delta < |a-1| < \delta^{-1}$. Thus (7.10) yields, for $x > x_4(\delta)$ and x on E,

$$I_a(e^x) < 2K\varepsilon\left(\log\frac{e}{\eta}\right)^{\alpha+|\beta|+1} \tag{7.11}$$

$$\times g(x)(\log g_0(x))^{-\alpha}(\log\log g_0(x))^{-\beta}\left(1+\log^+\frac{1}{h(x)}\right).$$

Next it is classical that if $\mu(r)$ is the maximum term and $N(r)$ the central index of $f(z)$, then

$$N(r) = r\frac{d}{dr}\log\mu(r) \tag{7.12}$$

except at isolated points, and $N(r)$ increases with r. Thus

$$N(r) \leqslant \frac{1}{h}\int_r^{re^h} N(t)\frac{dt}{t} \leqslant \frac{\log\mu(re^h)}{h} \leqslant \frac{\log M(re^h)}{h}.$$

Thus if $r = e^x$,

$$g_0(x) = \frac{\log M(re^h)}{h} > N(r).$$

On substituting this and (7.5) in (7.11) we deduce (7.2). This proves Theorem 8.

8. Functions of finite lower order

We can now prove our estimate when the order or lower order is finite.

THEOREM 9. *Suppose that $f(z)$ is an integral function of finite lower order satisfying the Fabry gap condition* (1.2). *Then there exists a set E, such that* *we have, as $r \to \infty$ on E, simultaneously*

$$\log L(r) \sim \log M(r) \tag{8.1}$$

and, if a is any complex number,

$$I_0(r, f_a) = o\{\log M(r,f)\}. \tag{8.2}$$

Further if $R \to \infty$, through any set of values satisfying for a fixed λ,

$$\log M(R,f) < R^{\lambda}, \tag{8.3}$$

we have

$$\int_{E(1,R)} \frac{dt}{t} \sim \log R. \tag{8.4}$$

COROLLARY 1. *E has upper logarithmic density 1, and if $f(z)$ has finite order, E has logarithmic density 1.*

COROLLARY 2. *We have (2.13) as $r \to \infty$ on E for $f_a(z)$ instead of $f(z)$, when a is fixed.*

COROLLARY 3. *The function $f(z)$ has no Borel exceptional values in any angle, i.e. the order of $N(r, \theta_1, \theta_2, a)$ is equal to that of $f(z)$ for any finite a, and $\theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$.*

Clearly Theorem 9 contains Theorem 3, when the order or lower order is finite. It is, however, a little more precise, since it tells us that E is 'thick' in those intervals $[1, R]$ for which $\log M(R)$ is not too large.

We proceed to prove Theorem 9. We apply Theorem 4 with $\eta_1 = \eta_2 = \lambda^{-1} = n^{-1}$, where n is a large positive integer, and let E_n be the set for which (4.12) holds. Then since $\Delta = 0$ in this case we deduce (4.12), i.e.

$$\log L(r,f) > \left(1 - \frac{1}{n}\right) \log M(r,f) \quad (r \in E_n), \tag{8.5}$$

where

$$\int_{E_n[1,R]} \frac{dt}{t} > \left(1 - \frac{2}{n}\right) \log R, \tag{8.6}$$

whenever R is sufficiently large and

$$\log M(R,f) < R^n. \tag{8.7}$$

Next it follows from Lemma 4, (4.5) applied with $T(r) = \log M(r)$, $K = \log 2$, that we have

$$\log M(re^h) < 2 \log M(r),$$

where $h = \log 2/n^2$, outside a set E_n', which also satisfies (8.6) when R is sufficiently large and satisfies (8.7). Thus outside E_n' we have

$$h(r) > \log 2/n^2, \tag{8.8}$$

where $h(r)$ is the quantity satisfying (7.3). Also we may apply Theorem 8, with $\alpha = \beta = 0$, and ε as small as we please for any fixed η. We chose $\eta = 1/n$, and ε so small that

$$K\varepsilon \log(en)(\log \tfrac{1}{2}n^2 + 1) = 1/n$$

in (7.2). Then we deduce that, if $r \in E$, where E is the set of that theorem and $h(r)$ satisfies (8.8), and if

$$a = 1, \quad \text{or} \quad n^{-1} < |1-a| < n, \tag{8.9}$$

and r is sufficiently large depending on n only, we have

$$I_a(r) < \frac{1}{n} \log M(r). \tag{8.10}$$

We deduce that we have simultaneously (8.5) and (8.10) outside a set F_n such that

$$\int_{F_n(1,R)} \frac{dt}{t} < \frac{6}{n} \log R, \tag{8.11}$$

whenever R satisfies (8.7), a satisfies (8.9), and $R > R_n$, say.

We assume that R_n is chosen to satisfy the above conditions, and, in addition,

$$R_n > R_{n-1}{}^{n-1}. \tag{8.12}$$

We now define

$$F = \bigcup_{n=1}^{\infty} F_n[R_n, R_{n+1}]. \tag{8.13}$$

Thus $r \in F$, if $R_n \leqslant r \leqslant R_{n+1}$ for some n and $r \in F_n$.

Suppose now that R satisfies (8.3) and $R > R_{n_0}$, where $n_0 = [\lambda]+2$.

Then we have $R_n \leqslant R \leqslant R_{n+1}$, where $n > \lambda + 1$; and, if $R_{n-1} < t < R$ and $t \in F$, we see that $t \in F_{n-1} \cup F_n = F_{n-1}$, so that, in view of (8.12), we have

$$\int_{F(1,R)} \frac{dt}{t} \leqslant \int_{F_{n-1}(1,R)} \frac{dt}{t} + \log R_{n-1} < \frac{6}{n-1} \log R + \frac{1}{n-1} \log R_n$$
$$< \frac{7}{n-1} \log R.$$

Thus if E is the complement of F, we see that (8.4) holds as $R \to \infty$ through any set of values satisfying (8.3) for a fixed λ. Since $f(z)$ has finite lower order λ_0 the set of R satisfying (8.3) is unbounded for $\lambda > \lambda_0$, so that E has upper logarithmic density 1. If $f(z)$ has finite order λ_0 and $\lambda > \lambda_0$, then (8.3) holds for all large R, so that E has logarithmic density 1.

Again if $r \in E$, $R_n \leqslant r \leqslant R_{n+1}$, and a satisfies (8.9), we see that r is outside F_n, and so (8.5) and (8.10) hold. In particular (8.1) and (8.2) hold as $r \to \infty$ in E if a is fixed. This completes the proof of Theorem 9.

It remains to prove the corollaries. We have already noted that (8.4), subject to (8.3), implies Corollary 1. Next Corollary 2 follows from Corollary 1 of Theorem 2, since if $f(z)$ satisfies (8.1), so do all the functions $f_a(z)$ for any fixed a.

It remains to prove Corollary 3. If $f(z)$ has zero order we have

$$N(r, \theta_1, \theta_2, a) \leqslant N(r, 0, 2\pi, a) \leqslant \log M(r) + O(1) = O(r^{\varepsilon})$$

as $r \to \infty$ for any $\varepsilon > 0$. Thus $N(r, \theta_1, \theta_2, a)$ has zero order.

We prove similarly that the order of $N(r, \theta_1, \theta_2, a)$ can never exceed that of $\log M(r)$. It remains to prove the opposite inequality, when $f(z)$ has positive order μ. Suppose first that $\mu < \infty$. Then given $\varepsilon > 0$, we can find r_n as large as we please such that

$$\log M(r_n) > r_n^{\mu(1-\varepsilon)}.$$

Also, in view of Corollary 1, we can for all sufficiently large n find $r'_n \in E$, such that

$$r_n < r'_n < r_n^{1+\varepsilon}.$$

Thus

$$\log M(r'_n) > \log M(r_n) > r_n^{\mu(1-\varepsilon)} > r'_n{}^{\mu(1-\varepsilon)/(1+\varepsilon)}.$$

If $n \to \infty$ through the sequence r'_n, and θ_1, θ_2, a are fixed, we deduce from Corollary 2 that

$$N(r'_n, \theta_1, \theta_2, a) > \left[\frac{\theta_2 - \theta_1}{2\pi} + o(1)\right] r'_n{}^{\mu(1-\varepsilon)/(1+\varepsilon)}.$$

Thus $N(r, \theta_1, \theta_2, a)$ has order at least $\mu(1-\varepsilon)/(1+\varepsilon)$, i.e. at least μ.

Finally, suppose that $\mu = \infty$, but that $f(z)$ has finite lower order λ_0. Choose λ_1, λ_2 so that $\lambda_0 < \lambda_1 < \lambda_2 < \infty$. Then there exist arbitrarily large values of $r = r'_n$, such that

$$\log M(r'_n) \leqslant r'_n{}^{\lambda_1}. \tag{8.14}$$

Also since $\mu = \infty$, there exist arbitrarily large values r_n of r, such that

$$\log M(r_n) \geqslant r_n^{\lambda_2}. \tag{8.15}$$

Given r'_n, we choose for r_n the largest number satisfying (8.15) and $r_n < r'_n$, which is possible for large n, since $r'_n \to \infty$ with n. Thus

$$\log M(r) \leqslant r^{\lambda_2}, \quad r_n \leqslant r \leqslant r'_n. \tag{8.16}$$

Also, since $\log M(r)$ increases with r, we deduce from (8.14) and (8.15) that

$$r_n'^{\lambda_1} \geqslant r_n^{\lambda_2}, \quad \text{i.e. } r_n' \geqslant r_n^{\lambda_2/\lambda_1}.$$

Thus it follows from (8.4) and (8.15) that we can find r_n'', such that

$$r_n \leqslant r_n'' \leqslant r_n^{\lambda_2/\lambda_1},$$

and $r_n'' \in E$, when n is large. Now Corollary 2 yields, for any fixed a, as $n \to \infty$

$$N(r_n'', \theta_1, \theta_2, a) \geqslant \left\{\frac{\theta_2-\theta_1}{2\pi}+o(1)\right\}\log M(r_n'')$$

$$\geqslant \frac{\theta_2-\theta_1}{4\pi} r_n^{\lambda_2} \geqslant \frac{\theta_2-\theta_1}{4\pi} r_n''^{\lambda_1}.$$

We can choose λ_1 as large as we please and deduce that $N(r, \theta_1, \theta_2, a)$ has infinite order. This completes the proof of Corollary 3.

9. Completion of proof of Theorem 3

By proving Theorem 9 we have proved the part of Theorem 3 which refers to functions of finite order or finite lower order. We now obtain a corresponding result for general functions under the gap condition (2.15). In view of Theorem 5 we can confine ourselves to estimating $I_a(r)$. In fact we can obtain a somewhat stronger result. This is

THEOREM 10. *Suppose that $f(z)$ is an integral function satisfying the gap-condition*

$$\varphi(n) = o\{n(\log n)^{-\alpha}(\log\log n)^{-\beta}\} \quad \textit{as} \quad n \to \infty, \tag{9.1}$$

where $\alpha > 0$, or $\alpha = 0$, $\beta > 1$. Then there exists a set E of logarithmic density 1, such that we have, for any fixed a, as $r \to \infty$ on E

$$I_a(r) = o(\log M(r))(\log N(r))^{-\alpha}(\log\log N(r))^{1-\beta}, \tag{9.2}$$

where $N(r)$ is the central index of $f(z)$.

We shall apply Theorem 8. Before doing so we shall, however, need to deal with $h(r)$. Our conclusion is contained in

LEMMA 11. *Suppose that $h(r)$ is a quantity satisfying* (7.3). *Then*

$$\log^+ \frac{1}{h(r)} < 2\log\log N(r), \tag{9.3}$$

outside a set of r of logarithmic density zero.

Suppose that E is the set where (9.3) is false. Let R be any number such that

$$\log M(R) \leqslant R. \tag{9.4}$$

Then if h_0 is any fixed constant, we have, for large r in E,

$$h(r) < h_0, \tag{9.5}$$

since $N(r) \to \infty$, with r. Thus it follows from (4.4) and (4.5) of Lemma 2 applied with $\lambda = 1$ and $K = \log 2$, and (7.3), that (9.5) can hold at most on a set E_0 of r such that

$$\int_{E_0(1,R)} \frac{dt}{t} < \frac{h_0}{\log 2} \log R + O(1),$$

as $R \to \infty$ for R satisfying (9.4). Since $h(r)$ satisfies (9.5) for all large r in E, we deduce that

$$\int_{E(1,R)} \frac{dt}{t} = o(\log R),$$

as $R \to \infty$ through values satisfying (9.4).

Next it follows from (4.3) of Lemma 4, applied with $T(r) = \log M(r)$, and (7.3) that

$$h(r) \geqslant \frac{4}{(\log\log M(r))^2} \tag{9.6}$$

outside a set of finite logarithmic measure. Also, if $\mu(r)$ is the maximum term we have as $r \to \infty$ outside a set of finite logarithmic measure (**[16]**),

$$\log \mu(r) \sim \log M(r).$$

Next in view of (7.12), we see that

$$\log \mu(r) = \int_1^r \frac{N(t)\,dt}{t} + O(1) \leqslant \log rN(r) + O(1).$$

Thus

$$N(r) \geqslant \frac{\log \mu(r)}{\log r} + o(1) > \frac{\log M(r)}{2 \log r}$$

outside a set of finite logarithmic measure. Thus if (9.4) is false and R is outside a set of finite logarithmic measure, we have (9.6) and

$$N(R) > (\log M(R))^{\frac{1}{2}},$$

which gives

$$\frac{1}{h(R)} < (\log N(R))^2.$$

Hence if R is sufficiently large this gives

$$\log^+ \frac{1}{h(R)} < 2 \log\log N(R). \tag{9.7}$$

If (9.4) is false for all large R, we deduce (9.7) for all large R outside a set of finite logarithmic measure, which proves Lemma 11. Otherwise, if R_1 is large, let R_2 be the largest number such that $R_2 < R_1$ and (9.4) holds for $R = R_2$. Then, for $R_2 \leqslant R \leqslant R_1$, we have (9.7) outside a fixed set E of finite logarithmic measure. Also for $1 \leqslant R \leqslant R_2$, we have (9.3) outside a set of logarithmic measure $o(\log R_2) = o(\log R_1)$. Thus we have in all cases, if E is the set where (9.3) is false,

$$\int_{E(1,R_2)} \frac{dt}{t} = o(\log R_2).$$

This proves Lemma 11.

9.1. We can now prove Theorem 10, and for this purpose we apply Theorem 8. We choose a positive integer p, and define ε by

$$3K\varepsilon \log(ep)^{\alpha+|\beta|+1} = \frac{1}{p}.$$

By hypothesis we have (7.1) for all large n. We deduce from (7.2) and Lemma 11 that we have, outside a set F_p of upper logarithmic density at most $\eta = p^{-1}$,

$$I_a(r) < \frac{1}{p}\left(\log^+ \frac{1}{h(r)} + 1\right) \log M(r)(\log N(r))^{-\alpha}(\log\log N(r))^{-\beta}$$

$$< \frac{3}{p} \log M(r)(\log N(r))^{-\alpha}(\log\log N(r))^{1-\beta}, \tag{9.8}$$

provided that

$$a = 1 \quad \text{or} \quad \frac{1}{p} < |a-1| < p. \tag{9.9}$$

In particular (9.8) holds, subject to (9.9), outside a set F_p, such that

$$\int_{F_p(1,R)} \frac{dt}{t} < \frac{2}{p} \log R \quad (R > R_p).$$

We suppose again that the R_p satisfy (8.12), define F by (8.13), and deduce as before that F has logarithmic density zero and that (9.8) holds, subject to (9.9), if r is outside F and $r > R_p$. This completes the proof of Theorem 10.

9.2. *Proof of Theorem* 3. We now suppose that $f(z)$ is an integral function satisfying (2.15). Then it follows from Theorem 5 that

$$\log L(r) \sim \log M(r) \tag{9.10}$$

as $r \to \infty$ on a set E_1 of logarithmic density 1, and from Theorem 10 that there exists a set E_2 of logarithmic density 1, such that for any fixed a

$$I_a(r) = o\{\log M(r)(\log N(r))^{-1}(\log\log N(r))^{1-\alpha}\} \tag{9.11}$$

as $r \to \infty$ in E_2. If $E = E_1 \cap E_2$ then (9.10) and (9.11) hold simultaneously as $r \to \infty$ in E, and E has logarithmic density 1. This completes the proof of Theorem 3.

In fact the argument gives a little more. If we apply (9.11) to $f_a(z)$, and note that

$$\log|f_a(z)| = \log|f(z)| + O(\log r),$$

we deduce from (2.10) that

$$N(r, \theta_1, \theta_2, a) = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \log|f(re^{i\theta})|\,d\theta + o\left\{\frac{\log M(r)}{(\log N(r))^{\frac{1}{2}}(\log\log N(r))^{\frac{1}{2}(\alpha-1)}}\right\}$$

as $r \to \infty$ on E. A corresponding conclusion also holds under the hypothesis (9.1) but if $\alpha < 1$, or $\alpha = 1$, $\beta < 2$, we cannot deduce (9.10). If (9.1) holds with $\alpha > 1$, or $\alpha = 1$, $\beta > 2$, we have (9.10) and (9.2) and we deduce from this and (2.10)

$$N(r, \theta_1, \theta_2, a) = \log M(r)\left[(1+o(1))\frac{\theta_2-\theta_1}{2\pi} + o(\log N(r))^{-\frac{1}{2}\alpha} \times (\log\log N(r))^{\frac{1}{2}(1-\beta)}\right]$$

uniformly for $\theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$. In particular

$$N(r, \theta_1, \theta_2, r) = (1+o(1))\log M(r)\frac{\theta_2-\theta_1}{2\pi},$$

provided that

$$\theta_2 - \theta_1 > \frac{1}{(\log N(r))^{\frac{1}{2}\alpha}(\log\log N(r))^{\frac{1}{2}(\beta-1)}}.$$

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