The first two problems should be quite easy if you know how to program already. If you don’t, then these problems will force you to use Matlab, Maple, or Sage, and learn a bit of coding. I encourage you to get help from your colleagues and me. In what follows, for any univariate polynomial \( f(x) = \sum_{i=1}^d c_i x^{x_i} \in \mathbb{C}[x] \setminus \{0\} \), with \( t \) monomial terms, we define its Archimedean Newton polygon to be
\[
\text{ArchNewt}(f) := \text{Conv}\left( \{ (a_i, -\log |c_i|) \}_{i \in \{1, \ldots, t\}} \right).
\]

**1:** Please write (in any language of your choice) a program to compute the convex hull of any input points \( p_1, \ldots, p_k \in \mathbb{R}^2 \). Extra points if you implement a method (e.g., specifying a tolerance a priori) for merging edges with nearly the same slope.

**2:** The Archimedean Newton polygon is due neither to Archimedes nor Newton: As far as I know, it first appeared in work of Jacques Hadamard around 1893. In this problem, you’ll use it to see the simplest, concrete connection between polynomials and polytopes.

(a) Let \( f(x) := \frac{1}{39} - x^{16} + x^{49} \). Please compute the vertices of \( \text{ArchNewt}(f) \), and the slopes of all the edges of \( \text{ArchNewt}(f) \).

(b) Continuing Part (a), what is the cardinality of \( \{ \log |\zeta| \mid f(\zeta) = 0, \zeta \in \mathbb{C} \setminus \{0\} \} \)?

(c) Continuing (b), verify computationally that, for any complex root \( \zeta \) of \( f \), there is an edge of \( \text{ArchNewt}(f) \) of slope \( s \) with \( | \log |\zeta| - s | < 0.001 \).

(d) Now let \( g(x) = 1 + x + \cdots + x^8 - x^9 + \frac{1}{2} x^{10} + \frac{1}{3} x^{11} + \cdots + \frac{1}{19} x^{18} \). Please find the least \( \varepsilon \) you can such that for any complex root \( \zeta \) of \( g \), there is an edge of \( \text{ArchNewt}(g) \) of slope \( s \) with \( | \log |\zeta| - s | < \varepsilon \).

**3:** (a) Let \( S \subseteq \mathbb{R}^n \) be any subset and let
\[
\mathcal{F}_{\geq}(S) := \{ f \in \mathbb{R}[x_1, \ldots, x_n] \mid f(x) \geq 0 \text{ for all } x \in S \}.
\]
Please prove that \( \mathcal{F}_{\geq}(S) \) is a cone, i.e., \( \mathcal{F}_{\geq}(S) \) is closed under nonnegative linear combinations.

(b) Please prove that any polyhedron is convex.

(c) Please prove that a univariate polynomial \( f \in \mathbb{R}[x] \) is nonnegative on all of \( \mathbb{R} \) \( \iff \) \( f \) is a sum of squares, i.e., \( f = g_1^2 + \cdots + g_k^2 \) identically, for some univariate polynomials \( g_1, \ldots, g_k \in \mathbb{R}[x] \). **Hint:** It helps if you use the Fundamental Theorem of Algebra to first deduce that \( f = c(x - \alpha_1) \cdots (x - \alpha_s) \) for some \( c, \alpha_1, \ldots, \alpha_s \in \mathbb{C} \), and then use the fact that all the coefficients of \( f \) are real to deduce further restrictions on \( c, \alpha_1, \ldots, \alpha_s \).

**4:** Let \( \{ a_1, \ldots, a_t \} \subseteq \mathbb{R}^n \), let \( A \) be the matrix with 1st column \( a_i \), and let \( \hat{A} \) be the matrix with i-th column \[
\begin{bmatrix}
1 \\
a_i
\end{bmatrix}.
\]
We call \( \{ a_1, \ldots, a_t \} \) **affinely independent** if and only if \( \hat{A} \) has right null-space \( \{ \mathbf{0} \} \).

(a) Please prove that \( A \) affinely independent \( \iff \) \( t - 1 \) is the dimension of the smallest affine subspace containing \( \{ a_1, \ldots, a_t \} \).

(b) Assume \( A \) is affinely independent with \( t = n + 1 \). Please prove that \( \alpha \in \mathbb{R}^n \) lies in the interior of the convex hull of \( \{ a_1, \ldots, a_t \} \) \( \iff \) there are \( \lambda_1, \ldots, \lambda_t > 0 \) with \( \lambda_1 a_1 + \cdots + \lambda_t a_t = \alpha \) and \( \lambda_1 + \cdots + \lambda_t = 1 \). (Recall that \( \alpha \in \mathbb{R}^n \) lies in the interior of a set \( S \subseteq \mathbb{R}^n \) if and only if there is an open ball \( B \) with \( \alpha \in B \subseteq S \).)

**5:** Please prove that, for any \( S \subseteq \mathbb{R}^n \), \( \text{Conv}(S) \) is the set of all convex combinations of finitely many elements of \( S \). **Note:** Recall that our original definition of \( \text{Conv}(S) \) was via the intersection of all convex sets containing \( S \). **Hint:** Let \( A := \text{Conv}(S) \) and let \( B \) denote the set of all convex combinations of finitely many elements of \( S \). The harder containment (of \( A \subseteq B \) and \( B \subseteq A \)) can be handled easily by induction.

**6:** (a) Let \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^{m \times 1} \). Please prove Fredholm’s Theorem of the Alternative: The linear system \( Ax = b \) has no solution in \( \mathbb{R}^n \) \( \iff \) there is a \( y \in \mathbb{R}^{1 \times n} \) such that \( yA = 0 \) and \( yb \neq 0 \). **Hint:** This is easy if you remember reduced row echelon form from basic linear algebra.

(b) Does Fredholm’s Theorem still hold if we replace \( \mathbb{R} \) above by an arbitrary field \( K \)? Please prove your answer.

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