Due Thursday, Sept. 22, 2011.

Problems

1: Please prove that the number of prime factors of a positive integer n is $O(\log n)$. **Hint:** If the primes dividing n are $p_1 < \cdots < p_k$ then we clearly have $n \geq p_1 \cdots p_k$. Can you then turn this into an upper bound on k?

2: Suppose $g \in \mathbb{C}[x_1, \ldots, x_N]$ is not identically zero and define $Z_{\mathbb{C}}(g) \subseteq \mathbb{C}^N$ to be the set of complex roots of g. Please show that $\mathbb{C}^N \setminus Z_{\mathbb{C}}(g)$ is path-connected. **Hint:** Try reducing to the special case $N = 1$, which you've already solved in HW#1. Perhaps there's a suitable parametrized line of the form $\{(a_1 t + b_1, \ldots, a_N t + b_N)\}_{t \in \mathbb{C}}$ to which you can restrict?

3: Please prove the following version of **Schwartz's Lemma**: If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a polynomial of degree d and $S \in \mathbb{C}$ is any subset of finite cardinality N, then f vanishes at at most dN^{n-1} points of S^n.
Hint: The $n = 1$ case follows immediately from the Weak Fundamental Theorem of Algebra. For general n, you can proceed by induction, and the key trick is to consider f as a polynomial in x_n with coefficients in $\mathbb{C}[x_1, \ldots, x_{n-1}]$.

Recall that given any $A \in \mathbb{Z}^{m \times n}$ you can always find a Hermite Factorization $UA = H$ by using integer *elementary row operations,*[1] *In particular, $U \in \mathbb{Z}^{m \times m}$ has determinant ± 1 and H is upper-triangular. When you have that H* also *satisfies $m \leq n$ and:*

(a) the left-most nonzero entry in each row of H is positive, and

(b) if $h_{i,j}$ is the left-most nonzero entry of row i, then $0 \leq h_{i',j} < h_{i,j}$ for all $i' < i$.

then the factorization is unique and we say that $UA = H$ is <u>the</u> Hermite Factorization of A (and H is the <u>the</u> Hermite Normal Form of A).

4: **(a)** Please find the Hermite Factorizations for the following matrices:

$$\begin{bmatrix} 6 \\ 15 \\ 10 \end{bmatrix}, \begin{bmatrix} 27 & -11 \\ 3 & 21 \end{bmatrix}, \begin{bmatrix} 6 & 7 & 11 \\ 2 & 4 & -8 \end{bmatrix}.$$

(b) Suppose now that $A \in \mathbb{Z}^{n \times n}$, $\det A \neq 0$, and $c = (c_1, \ldots, c_n)$ with $c_1, \ldots, c_n \in \mathbb{C}^*$. Please find the most elegant formula you can for the number of roots of the $n \times n$ binomial system

$$x^A = c.$$

Hint: You can actually state an elegant formula without the Hermite Factorization, but Hermite Factorization can greatly simplify the proof of your formula.

(c) Following the notation of Part (b), suppose instead that $\det A = 0$. Please characterize, via the Hermite Factorization, exactly when the $n \times n$ binomial system

$$x^A = c$$

has no (resp. infinitely many) solutions.

Refining Hermite normal form, a **Smith** *factorization is an expression of the form $S = UAV$, with S nonnegative and* **diagonal***, $U \in \mathbb{Z}^{m \times m}$, and $V \in \mathbb{Z}^{n \times n}$ with $|\det U| = |\det V| = 1$. If $S = [s_{i,j}]$, $m \leq n$, and, in addition to the preceding conditions, $s_{i,i} | s_{i+1,i+1}$ for all i, then such*

[1] i.e., the only scaling of a row allowed is multiplication by ± 1.

a factorization is unique and is called **the** *Smith factorization. We also call* S **the Smith Normal Form (SNF)** *of* A.

5: Please find the most elegant explicit formula you can for all the roots of the following system of equations:

$$x^{27}y^3 = a$$
$$x^{-11}y^{21} = b,$$

where a and b are nonzero complex constants. **Hint:** Here, the Smith Normal Form will prove quite useful.

An **open orthant in** $\mathbb{R}^n$ *is any reflection, across one or more coordinate hyperplanes, of the positive orthant in* $\mathbb{R}^n$. *(So* $(\mathbb{R}^*)^n$ *is the disjoint union of exactly* 2^n *open orthants.) A function* $F : (\mathbb{R}^*)^n \longrightarrow (\mathbb{R}^*)^n$ *is said to be* N**-to-**1 *iff* $\#F^{-1}(y) = N$ *for all* y *in the range of* F.

6: Given any invertible $n \times n$ matrix $A \in \mathbb{Z}^{n \times n}$, show that the function $F : (\mathbb{R}^*)^n \longrightarrow (\mathbb{R}^*)^n$ defined by $F(x) := x^A$ is N-to-1, and show how to compute N — and the range of F — explicitly via the Smith normal form of A. **Hint:** Note that the special case $n = 1$ depends on the parity (oddness/even-ness) of the single entry of A. Then, try to show that the special case where A is unimodular yields $N = 1$ and go from there.

NOTE: Please feel free to e-mail comments, questions, and/or corrections.