Three of these problems will be graded (you won’t know which), but you should know how to do them all. Please note that the problems marked $H$ are optional if you are in the regular section, but are required if you are in the honors section.

1: Please find bases for the right nullspaces of the matrices below:

(a) $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 0 & 1 & 0 & 2 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

2: Please prove that if $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$ are bases for a vector space $V$ then $m=n$.

3: Let $F$ be any field. Recall that $F[x]$ denotes the set of polynomials with coefficients in $F$ and the degree of $f$ is the largest exponent appearing in $f$. Please decide whether the following sets are vector spaces (with respect to polynomial addition):

(a) $\{0\} \cup \{f \in F[x] \mid f \text{ has degree divisible by 3}\}$
(b) $\{0\} \cup \{f \in F[x] \mid f \text{ has degree divisible by 17}\}$

4: Let us define a function $T : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ as follows: $T(f)(x)$ is defined to be $\frac{f(x)-f(7)}{x-7}$ or $f'(7)$, according as $x \neq 7$ or $x = 7$. Please prove that $T$ is a linear map.

5: Recall that, for any field $F$, any vector space $V$ over $F$, and any linear map $T : V \rightarrow V$, we call $(\lambda, v) \in F \times V$ an eigenpair if and only if $Tv = \lambda v$. (We also call $\lambda$ the eigenvalue associated with the eigenvector $v$.)

(a) Please find all eigenpairs for the linear map $\frac{d}{dx} : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$.
(b) Please give an example of a linear map with no eigenvalues. **Hint:** There are numerous examples if you use a field other than $\mathbb{C}$.
(c) Suppose $T : V \rightarrow V$ is a linear map and $v, w \in V$ satisfy $T(v) = 4w$ and $T(w) = 4v$. Prove that either $4$ or $-4$ is an eigenvalue of $T$.

6: Suppose $v \in \mathbb{R}^n$ is a column vector with transpose $v^\top$. Recall that an invariant subspace of a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the set of all eigenvectors associated to a fixed eigenvalue.

(a) Please find all eigenvalues and invariant subspaces for the matrix $vv^\top/v^\top v$. **Hint:** First find a basis for the right nullspace of $v^\top$.
(b) Let $I_n$ denote the $n \times n$ identity matrix. Please find all eigenvalues and invariant subspaces for the matrix $I_n - vv^\top/v^\top v$. **Hint:** If you know the image of a suitable basis of $\mathbb{R}^n$ under $vv^\top/v^\top v$, it is easy to find the image of the basis under $I_n - vv^\top/v^\top v$, and then the problem is easy.

7$H$: (a) Please prove that our algorithmic definition of the determinant makes sense for upper-triangular matrices, i.e., no matter what order we apply elementary row operations to $A$ to reduce to row echelon form, we still get the same value for the determinant.

(b) For any subset $\{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$ of cardinality $d$, let $(i_1 \cdots i_d)$ denote the permutation of $\{1, \ldots, n\}$ that sends $i_j$ to $i_{j+1}$ (for all $j \in \{1, \ldots, j-1\}$) and $i_d$ to $i_1$. We call $(i_1i_2)$ a transposition. Please prove that if $(n \cdots 1)$ is written as a composition of transpositions, the number of transpositions must be odd or even according as $n$ is even or odd.

**NOTE:** Please feel free to e-mail comments, questions, and/or corrections.

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