1. Introduction

One of the happiest coincidences in algebraic geometry is that the norms of roots of polynomials can be estimated through polyhedral geometry. Perhaps the earliest incarnation of this fact was Isaac Newton’s use of a polygon to determine series expansions for algebraic functions. This was detailed in a letter, dated October 24, 1676 [New76], that Newton wrote to Henry Oldenburg. In modern terminology, Newton computed (s-adic) norms of roots of univariate polynomials over the Puiseux series field \( \mathbb{C}((s)) \), i.e., the union of formal Laurent series fields \( \bigcup_{d \in \mathbb{N}} \mathbb{C}((s^{1/d})) \).

Definition 1.1. Let \([N] := \{1, \ldots, N\} \). We define the s-adic valuation of any element \( \zeta = \sum_{j=k} c_j s^{j/d} \) of \( \mathbb{C}((s)) \) to be \( \text{ord}_s \zeta := \min_{c_j \neq 0} j/d \). (We set \( \text{ord}_s 0 := \infty \) and thus \( \text{ord}_s : \mathbb{C}((s)) \to \mathbb{Q} \cup \{\infty\} \).) We also let \( \text{Conv}(U) \) denote the convex hull of (i.e., smallest convex set containing) a set \( U \subseteq \mathbb{R}^n \). For any \( f \in \mathbb{C}((s))[x_1] \) written \( f(x_1) = \sum_{i=1}^t c_i x_1^{a_i} \), with \( t \geq 2, a_1 < \cdots < a_t \), and \( c_i \neq 0 \) for all \( i \), we define its s-adic Newton polygon to be \( \text{Newt}_s(f) := \text{Conv}(\{ (a_i, \text{ord}_s c_i) \mid i \in [t] \}) \). Finally, we define the (s-adic) tropical variety of \( f \) to be \( \text{Trop}_s(f) := \{ v \in \mathbb{R}^{|(v, 1)|} \text{ is an inner normal of an edge of } \text{Newt}_s(f) \} \). ⊗

Example 1.2. The trinomial \( f(x_1) = s - x_1^{16} + x_1^{49} \) has exactly 49 roots in \( \mathbb{C}((s)) \): 16 of the form \( e^{2\pi \sqrt{-1} k/16} s^{1/16} + \sum_{j=2}^{\infty} \alpha_{j,k} s^{j/16} \) (for \( k \in [16] \)) and 33 of the form \( e^{2\pi \sqrt{-1} k/33} + \sum_{j=1}^{\infty} \beta_{j,k} s^j \) (for \( k \in [33] \)), where \( \alpha_{j,k} \in \mathbb{Q} \left[ e^{2\pi \sqrt{-1} /16} \right] \) and \( \beta_{j,k} \in \mathbb{Q} \left[ e^{2\pi \sqrt{-1} /33} \right] \). Newton’s technique from [New76], in more recent language, gives us the initial exponents 1/16 and 0 exactly as the points of \( \text{Trop}_s(f) \). In particular, \( \text{Newt}_s(f) \) here is the convex hull of \( \{(0, 1/16), (16, 0), (49, 0)\} \), which is the triangle drawn below, along with 3 representative inner normals:

There are just two upward-pointing inner normals, and thus just two inner normals of the form \( (v, 1) \): \((1/16, 1)\) and \((0, 1)\). So \( \text{Trop}_s(f) = \{1/16, 0\} \) and this is exactly the set of values of \( \text{ord}_s \zeta \) over all roots \( \zeta \in \mathbb{C}((s)) \setminus \{0\} \) of \( f \). ⊗
Newton’s result has since been extended to other fields, such as algebraic extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ (see, e.g., [Dum06, Wei63]). Tropical geometry (see, e.g., [EKL06, LS09, IMS09, BR10, MS15]) continues to deepen the links between algebraic, arithmetic, and polyhedral geometry. However, finding an analogous approach for roots in $\mathbb{C}$ presents a metric complication: Unlike $\mathbb{C}$, each field $\mathbb{C}\langle\!\langle s\rangle\!\rangle$, $\mathbb{Q}_p$, and $\mathbb{F}_p((t))$ is endowed with a (non-trivial) non-Archimedean norm, i.e., a norm which is bounded on the embedded copy of $\mathbb{Z}$ in the respective underlying field. For instance, one can set $|\zeta|_s := e^{-\text{ord}_\zeta}$ for any $\zeta \in \mathbb{C}\langle\!\langle s\rangle\!\rangle$ and easily prove that $|\cdot|_s$ satisfies the Triangle Inequality, as well as the stronger Ultrametric Inequality $|x+y|_s \leq \max\{|x|_s, |y|_s\}$. In particular, $|Z|_s = \{0\}$.

Newton’s observations on Puiseux series imply that the number of possible $s$-adic norms, for the roots of a univariate polynomial with exactly $t$ terms, is at most $t - 1$. This fails for the usual (Archimedean) norm $|\cdot|$ on $\mathbb{C}$. However, with some care, we can still study Archimedean norms of roots of polynomials in a polyhedral/tropical way: Jacques Hadamard was possibly the first to define an analogue of Newt$_s$ for the usual norm on $\mathbb{C}$ [Had93] (see also [Ost40a] and [Val54, Ch. IX, pp. 193–202]). Here, we formulate a version applicable in arbitrary dimension. (See also [Mik04, PR04, PRS11, TdW13] for important precursors.)

**Definition 1.3.** We call any $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of the form $f(x) = \sum_{i=1}^t c_i x_i^{a_i}$, with $c_i \neq 0$ for all $i$ and $(a_1, \ldots, a_t)$ of cardinality $t$, an $n$-variate $t$-nomial. (The notation $x = (x_1, \ldots, x_n)$ and $x^n = x_1^{a_1} \cdots x_n^{a_n}$ is understood.) We then define the (ordinary) Newton polytope of $f$ to be $\text{Newt}(f) := \text{Conv}\{(a_i)_{i \in [t]}\}$, and the Archimedean Newton polytope of $f$ to be $\text{ArchNewt}(f) := \text{Conv}\{(a_i, -\log|c_i|)_{i \in [t]}\}$. We also define the Archimedean tropical variety of $f$ (provided $t \geq 2$) to be $\text{ArchTrop}(f) := \{w \in \mathbb{R}^n \mid \langle v, -1 \rangle \text{ is an outer normal of a positive-dimensional face of } \text{ArchNewt}(f)\}$. \(\diamond\)

**Example 1.4.** It is easily checked that for any univariate binomial $f$, $\text{ArchTrop}(f)$ is a single point in $\mathbb{R}$ and all the complex roots of $f$ lie on a circle of radius $e^{\text{ArchTrop}(f)}$ centered at the origin. More generally, for any $n$-variate binomial, $\text{ArchTrop}(f)$ is an affine hyperplane in $\mathbb{R}^n$ which is exactly the image of the complex roots of $f$ under the coordinate-wise log-absolute value map. \(\diamond\)

The norms of complex roots are not always described exactly by $\text{ArchTrop}(f)$, but $\text{ArchTrop}(f)$ nevertheless provides an approximation within an explicit tolerance.

**Example 1.5.** For a root $\zeta \in \mathbb{C}$ of $f(x_1) := \frac{1}{89} - x_1^{16} + x_1^{49}$ there are exactly 26 possible values for $|\zeta|$. However, these norms cluster tightly about just 2 values: Exactly 16 roots have norm approximately $89^{-1/16}$ (to at least 4 decimal places) and exactly 33 roots have norm approximately 1 (to 3 decimal places). Here, $\text{ArchNewt}(f)$ is the convex hull of $\{(0, -\log\frac{1}{89}), (16, 0), (49, 0)\}$, which is the triangle with outer normals as shown below:

![Diagram](image.png)

and $(0, -1)$. So $\text{ArchTrop}(f) = \{\log(89^{-1/16}), 0\}$ and the logs of the absolute values of the roots of $f$ in $\mathbb{C}$ indeed cluster tightly about $\text{ArchTrop}(f)$. \(\diamond\)

**Theorem 1.6.** For any univariate $t$-nomial $f$ with root $\zeta \in \mathbb{C} \setminus \{0\}$, we have that $\log |\zeta|$ lies in the union of open intervals $\bigcup_{v \in \text{ArchTrop}(f)} (v - \log 2, v + \log 2)$, or $\bigcup_{v \in \text{ArchTrop}(f)} (v - \log 3, v + \log 3)$, according as $t = 3$ or $t \geq 4$. In particular, $\#\text{ArchTrop}(f) \leq t - 1$ for such $f$. \(\diamond\)
We prove Theorem 1.6 in Section 2.1. Theorem 1.6 improves an earlier bound of Ostrowski [Ost40a, Bound (25, 3), pg. 145] which, letting $d$ denote the degree of $f$, implies that $\log |\zeta|$ lies in the larger union $\bigcup_{v \in \text{ArchTrop}(f)} (v - \log(d + 1), v + \log(d + 1))$.

It is also the case that, for any $v \in \text{ArchTrop}(f)$, there actually exists a root $\zeta \in \mathbb{C}$ of $f$ with $\log |\zeta|$ close to $v$. In particular, the clustering of $\text{ArchTrop}(f)$ determines certain annuli guaranteed to contain a positive number of roots of $f$. In what follows, for any line segment $L \subset \mathbb{R}^2$ with vertices $(a, b)$ and $(c, d)$, we define its horizontal length to be $\lambda(L) := |c - a|$.

**Theorem 1.7.** Given any univariate $t$-nomial $f$ with $t \geq 3$, let $\Gamma$ be any connected component of the union of open intervals $\bigcup_{v \in \text{ArchTrop}(f)} (v - \log 3, v + \log 3)$ and let $\Lambda_{\Gamma}$ be the sum of $\lambda(L)$ over all edges $L$ of $\text{ArchNewt}(f)$ with outer normal $(v, -1)$ and $v \in \Gamma$. Then the number of roots $\zeta \in \mathbb{C}$ of $f$ with $\log |\zeta| \in \Gamma$, counting multiplicity, is exactly $\Lambda_{\Gamma}$. In particular, $\Lambda_{\Gamma} \geq 1$.

Theorem 1.7 is proved in Section 2.2.

We can in fact polyhedrally approximate complex root norms in arbitrary dimension.

**Definition 1.8.** Let us set $\text{Log}|x| := (\log |x_1|, \ldots, \log |x_n|)$ and, for any $f \in \mathbb{C}[x_1^\pm 1, \ldots, x_n^\pm 1]$, define Amoeba$(f)$ to be $\{\text{Log}|x| \mid f(x) = 0, \ x \in (\mathbb{C} \setminus \{0\})^n\}$. ∗

**Example 1.9.** Taking $f(x) = 1 + x_1^3 + x_2^3 - 10x_1x_2$, it is easily checked that Newt$(f)$ is a triangle, while ArchNewt$(f)$ is a pyramid. In particular, ArchTrop$(f)$ is a polyhedral complex consisting of 3 vertices and 6 edges (3 of which are unbounded rays). An illustration of Amoeba$(f) \cap [-7, 7]^2$ and ArchTrop$(f) \cap [-7, 7]^2$ appears to the right. Amoeba$(f)$ is lightly shaded and contains ArchTrop$(f)$ (drawn darker). ∗

Our main result is that every point of Amoeba$(f)$ is within an explicit distance of some point of ArchTrop$(f)$, and vice-versa, independent of the degree or number of variables of $f$. We use $\cdot |$ for the standard $\ell_2$-norm on $\mathbb{C}^n$.

**Definition 1.10.** For any $\varepsilon > 0$ and $X \subseteq \mathbb{R}^n$ we define the (open) $\varepsilon$-neighborhood of $X$ to be $X_\varepsilon := \{x \in \mathbb{R}^n \mid \|x - x'\| < \varepsilon \text{ for some } x' \in X\}$. ∗

**Theorem 1.11.** For any $f \in \mathbb{C}[x_1^\pm 1, \ldots, x_n^\pm 1]$ with exactly $t \geq 2$ monomial terms and Newt$(f)$ of dimension $k$ we have $t \geq k + 1$ and:

1. For $k = 1$ we have that ArchTrop$(f)$ is a non-empty disjoint union of at most $t - 1$ parallel affine hyperplanes in $\mathbb{R}^n$, while for $k \geq 2$ we have that ArchTrop$(f)$ is a path-connected $(n - 1)$-dimensional polyhedral complex with at most $t(t - 1)/2$ faces of dimension $n - 1$.
2. For $t = k + 1$ we have ArchTrop$(f) \subseteq \text{Amoeba}(f)$ and both Amoeba$(f)$ and ArchTrop$(f)$ are contractible.
3. For all $t \geq 3$ we have (a) Amoeba$(f) \subset \text{ArchTrop}(f)_{\varepsilon_1}$ and (b) ArchTrop$(f) \subset \text{Amoeba}(f)_{\varepsilon_2}$, where $\varepsilon_1 := \log(t - 1)$ and $\varepsilon_2 := (\log 9)t - \log \frac{81}{2} < 2.2t - 3.7$.
4. Let $\varphi(x) := 1 + x_1 + \cdots + x_{t-1}$ and $\psi(x) := (x_1 + 1)^{t-k} + x_2 + \cdots + x_k$. Then (a) Amoeba$(\varphi)$ contains a point at distance $\log(t - 1)$ from ArchTrop$(\varphi)$ and (b) ArchTrop$(\psi)$ contains points approaching distance $\log(t - k)$ from Amoeba$(\psi)$. ∗
We prove Theorem 1.11 in Section 3. Note that Assertion (1) contains the special case where the degree of \( f \) is 1. Note also that Theorem 1.6 is a refinement of the special case \( n = 1 \) of Assertion (2a) above. (Indeed, when \( n = 1 \), \( \text{ArchTrop}(f)_{\varepsilon_1} \) is simply a finite union of open intervals of width \( 2\varepsilon_1 \.) We will see later in Section 3.4 how Theorem 1.7 implies Assertion (2b). Assertion (3a) of Theorem 1.11 shows that the size of the neighborhood from Assertion (2a) is in fact optimal.

For multivariate polynomials, our bounds appear to be the first allowing dependence on just the number of terms \( t \). In particular, letting \( L \) denote the number of lattice points in the Newton polytope of \( f \), Mikhalkin proved that \( \text{Amoeba}(f) \) is contained in the possibly larger neighborhood \( \text{ArchTrop}(f)_{\log(L-1)} \), in the special case \( n = 2 \) [Mik05, Lemma 8.5, pg. 360]. As far as we are aware, the only other earlier distance bound between an amoeba and a polyhedral approximation is a result of Viro [Vir01, Sec. 1.5] on the distance between the graph of a univariate polynomial (drawn on log paper) and a piece-wise linear curve that is ultimately a piece of the \( n = 2 \) case of \( \text{ArchTrop}(f) \) here.

**Example 1.12.** Setting \( \psi(x) = (x_1 + 1)^4 + x_2 \) we see \( \text{Amoeba}(\psi) \cap ([−7, 7] \times [−12, 12]) \) and \( \text{ArchTrop}(\psi) \cap ([−7, 7] \times [−12, 12]) \) on the right. \( \text{ArchTrop}(\psi) \) contains the ray \((\log 4, 4 \log 4) + \mathbb{R}_+(0, −1)\) and this rightmost downward-pointing ray contains points with distance from \( \text{Amoeba}(\psi) \) approaching \( \log 4 \). We also observe that Viro’s earlier polygonal approximation of graphs of univariate polynomials on log paper, applied here, would result in the polygonal curve that is the subcomplex of \( \text{ArchTrop}(\psi) \) obtained by deleting all 4 downward-pointing rays.

It is worth comparing Theorem 1.11 to two other methods for approximating complex amoebae: Purbhoo, in [Pur08], describes a uniformly convergent sequence of outer polyhedral approximations to any amoeba, using cyclic resultants. While \( \text{ArchTrop}(f) \) lacks this refinability, the computation of \( \text{ArchTrop}(f) \) is considerably simpler: see Section 1.2 below and [AGGR13]. \( \text{ArchTrop}(f) \) is actually closer in spirit to the spine of \( \text{Amoeba}(f) \). The latter construction, based on a multivariate version of Jensen’s Formula from complex analysis, is due to Passare and Rullgård [PR04, Sec. 3] and results in a polyhedral complex that is always contained in, and is homotopy equivalent to, \( \text{Amoeba}(f) \). Unfortunately, the computational complexity of the spine is not as straightforward as that of \( \text{ArchTrop}(f) \). Further background on the computational complexity of amoebae can be found in [The02, SdW13, TdW15].

Our final main results concern the complexity of deciding whether a given point lies in a given amoeba or Archimedean tropical variety. However, let first us observe a consequence of our metric estimates for systems of polynomials.

**1.1. Coarse, but Fast, Isolation of Roots of Polynomial Systems.** An immediate consequence of Assertion (2a) of Theorem 1.11 is an estimate for isolating the possible norm vectors of complex roots of arbitrary systems of multivariate polynomial equations.

**Corollary 1.13.** Suppose \( f_1, \ldots, f_k \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) where \( f_i \) has exactly \( t_i \) monomial terms for all \( i \). Then any root \( \zeta \in (\mathbb{C}^*)^n \) of \( F = (f_1, \ldots, f_k) \) satisfies

\[
\text{Log}(|\zeta|) \in \mathbb{Z}^{n}\cap \bigcap_{i=1}^k \text{ArchTrop}(f_i)_{\varepsilon_i},
\]

where \( \varepsilon_i := \log(t_i - 1) \) for all \( i \).

**Example 1.14.** We can isolate the log-norm vectors of the complex roots of the \( 3 \times 3 \) system

\[
F := (f_1, f_2, f_3) := (x_1 x_2 - x_1^2 - 1/16^6, x_2 x_3 - 1 - x_2^2/16^6, x_3 - 1 - x_3^2/16^18)
\]

via Corollary 1.13 as follows: find the points of \( X := \text{ArchTrop}(f_1) \cap \text{ArchTrop}(f_2) \cap \text{ArchTrop}(f_3) \).
by searching through suitable triplets of edges of the ArchNewt($f_i$), and then create isolating parallelepipeds about the points of $X$. More precisely, observe that $\text{Conv}((1, 1, 0, 0), (2, 0, 0, 0))$, $\text{Conv}((0, 1, 1, 0), (0, 0, 0, 0))$, and $\text{Conv}((0, 0, 1, 0), (0, 0, 0, 0))$ are respective edges of ArchNewt($f_1$), ArchNewt($f_2$), and ArchNewt($f_3$), and the vector $(0, 0, 0, -1)$ is an outer normal to each of these edges. So $(0, 0, 0)$ is a point of $X$. Running through the remaining triplets we then obtain that $X$ in fact consists of exactly 4 points:

$\log |(1/9, 1, 1)|$, $\log |(1, 1, 1)|$, $\log |(16^6, 16, 1)|$, and $\log |(16^{12}, 16^{12}, 16)|$.

So Corollary 1.13 tells us that the points of $Y := \text{Amoeba}(f_1) \cap \text{Amoeba}(f_2) \cap \text{Amoeba}(f_3)$ lie in the union of the 4 parallelepipeds drawn below to the right: Truncations of ArchTrop($f_1$), ArchTrop($f_2$), and ArchTrop($f_3$) are drawn below on the left, and the middle illustration uses transparency to further detail the intersection.

Suitably ordered, each point of $X$ is actually within distance $0.11 \times 10^{-6} (< 0.693... = \log 2)$ of some point of $Y$ (and vice-versa), well in accordance with Corollary 1.13. ∗

See [PR13] for the relevance of the preceding system to fewnomial theory over general local fields.

1.2. On the Computational Complexity of ArchTrop($f$) and Amoeba($f$). The complexity classes $\mathsf{P}$, $\mathsf{NP}$, $\mathsf{PSPACE}$, and $\mathsf{EXPTIME}$ — from the classical Turing model of computation — can be identified with families of decision problems, i.e., problems with a yes or no answer. Larger complexity classes correspond to problems with larger worst-case complexity. We refer the reader to [Sip92, Pap95, AB09, Sip12] for further background.

Aside from the basic definitions of input size and $\mathsf{NP}$-hardness, it will suffice here to simply recall that $\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXPTIME}$, and that the properness of each inclusion (save $\mathsf{P} \subseteq \mathsf{EXPTIME}$, which is well-known) is a famous open problem. All algorithmic complexity results below count bit operations, and do so as a function of some underlying notion of input size.

Deciding membership in an amoeba can easily be rephrased as a problem within the Existential Theory of the Reals. The latter setting has been studied extensively in the 20th century (see, e.g., [Tar51, CG84, Can88]) and the current state of the art implies that amoeba membership can be solved efficiently by a parallel algorithm. More precisely, we define the input size of a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, written $f(x) = \sum_{i=1}^{t} c_i x_{a_i}$, to be $\text{size}(f) := \sum_{i=1}^{t} \log \left( (2 + |c_i|) \prod_{j=1}^{n} (2 + |a_{i,j}|) \right)$, where $a_i = (a_{i,1}, \ldots, a_{i,n})$ for all $i$. (Put another way, up to a constant multiple, $\text{size}(f)$ is just the sum of the bit-sizes of all the coefficients and exponents.) Similarly, we define $\text{size}(v)$, for any $v = (v_1, \ldots, v_n) \in \mathbb{Q}^n$, to be the sum of the bit-sizes of the numerators and denominators of the $v_i$ (written in lowest terms). We similarly extend the notion of input size to polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$. Considering real and imaginary parts, we can extend further still to polynomials in $\mathbb{Q}[\sqrt{-1}][x_1, \ldots, x_n]$. 


Theorem 1.15. There is a PSPACE algorithm to decide, for any input pair \((v, f) \in \bigcup_{n \in \mathbb{N}} (\mathbb{Q}^n \times \mathbb{Q}[\sqrt{-1}][x_1, \ldots, x_n])\), whether \(\log|v|\) lies in Amoeba\((f)\). Furthermore, the special case where \(v = 1\) and \(f \in \mathbb{Z}[x_1]\) in the preceding membership problem is already NP-hard.

Theorem 1.15 is implicit in earlier work of Chistov and Grigoriev [CG84, Can88], and Plaisted [Pla84], so for the convenience of the reader, we sketch a proof in Section 2.3.

Remark 1.16. For our notion of input size, polynomial-time for sufficiently sparse polynomials implies polynomiality in the logarithm of the degree of the polynomial. This is in contrast to a looser notion of input size used by Theobald in [The02, Cor. 2.7], allowing “polynomial-time” point membership detection for amoebae in fixed dimension: The methods there yield complexity polynomial in the degree when \(n\) is fixed, thus yielding exponential complexity relative to the input size we use here. The NP-hardness lower bound from Theorem 1.15 suggests that speeding up point membership for amoebae to polynomial-time (relative to our notion of input size here) may not be possible.

Since we now know that ArchTrop\((f)\) is provably close to Amoeba\((f)\), ArchTrop\((f)\) would be of great practical value if ArchTrop\((f)\) were easier to work with than Amoeba\((f)\). This indeed appears to be the case. For example, when the dimension \(n\) is fixed and all the coefficient absolute values of \(f\) have rational logarithms, standard high-dimensional convex hull algorithms (see, e.g., [Ede87]) enable us to describe every face of ArchTrop\((f)\), as an explicit intersection of half-spaces, in polynomial-time.

The case of rational coefficients presents some subtleties because the underlying computations, done naively, involve arithmetic on rational numbers with exponentially large bit-size. Nevertheless, point membership for ArchTrop\((f)\) can be decided in polynomial-time when \(n\) is fixed.

Theorem 1.17. Suppose \(v = (v_1, \ldots, v_n) \in \mathbb{Q}^n\), and \(f \in \mathbb{Q}[\sqrt{-1}][x_1, \ldots, x_n]\) (written \(f(x) = \sum_{i=1}^t c_i x^{a_i}\)) has exactly \(t\) monomial terms, degree at most \(d\) with respect to any variable, and the bit-sizes of the \(v_i\) and \(c_i\) are at most \(\sigma\). Then there is an \(O(n t \log d + o(n t \log d))\) algorithm to decide, for any such input pair \((v, f)\), whether \(\log|v|\) lies in ArchTrop\((f)\).

Furthermore, if we instead assume that \(\log|c_i| \in \mathbb{Q}\) has bit size \(\leq \sigma\) for all \(i\), then there is an \(O(n t \sigma \log d \log^2(\sigma d))\) algorithm to decide whether \(v\) lies in ArchTrop\((f)\).

We prove Theorem 1.17 in Section 4.

Since ArchTrop\((f)\) has codimension 1 as a semi-algebraic subset of \(\mathbb{R}^n\), it is rather unlikely in practice for a random query point \(v \in \mathbb{R}^n\) to lie in ArchTrop\((f)\). So a more practical problem would be to efficiently describe which connected component of \(\mathbb{R}^n \setminus \text{ArchTrop}(f)\) contains a query point \(v\), or to find the point of ArchTrop\((f)\) nearest to \(v\). The complexity of these problems, as well as their relevance to polynomial system solving, is explored further in [AGGR13].

1.3. Ideas Behind the Proofs and a Useful Hausdorff Limit. A key idea behind our metric results is the following fact: Knowing where a polynomial \(f\) has exactly 2 monomials of largest norm is enough to recover useful information about the complex roots of \(f\). In particular, it is easy to show that ArchTrop\((f)\) admits the following alternative characterization.

Proposition 1.18. For any \(n\)-variate \(t\)-nomial \(f(x) = \sum_{i=1}^t c_i x^{a_i}\) we have

\[
\text{ArchTrop}(f) = \left\{ v \in \mathbb{R}^n \mid \max_i |c_i e^{a_i} \cdot v| \text{ is attained for at least two distinct values of } i \right\}.
\]
Remark 1.19. We adopt the natural conventions \( \text{Trop}_s(0) = \text{ArchTrop}(0) = \mathbb{R}^n \) and \( \text{Trop}_s(cx^n) = \text{ArchTrop}(cx^n) = \emptyset \) for any \( c \neq 0 \) and \( a \in \mathbb{Z}^n \). ◦

It is also conceptually important to recall a non-Archimedean precursor to our main result: Letting \( \text{ord}_s x := (\text{ord}_s x_1, \ldots, \text{ord}_s x_n) \) for any \( x = (x_1, \ldots, x_n) \in \mathbb{C} \langle \langle s \rangle \rangle^n \), and making the natural extension \( \text{Trop}_s(f) = \{ v \in \mathbb{R}^n \mid (v, 1) \text{ is an inner normal of a face of } \text{Newt}_s(f) \text{ of positive dimension} \} \), the statement is as follows:

Kapranov’s Non-Archimedean Amoeba Theorem. (Special case) [EKL06] For any \( f \in \mathbb{C} \langle \langle s \rangle \rangle [x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), we have \( \{ \text{ord}_s \zeta \mid f(\zeta) = 0 \text{ and } \zeta \in (\mathbb{C} \langle \langle s \rangle \rangle \setminus \{ 0 \})^n \} = \text{Trop}_s(f) \cap \mathbb{Q}^n \). ■

Kapranov’s result above (derived no later than 2000) is to our Theorem 1.11 as Newton’s Puisieux series characterization [New76] is to our Theorem 1.6.

In Kapranov’s Theorem, the containment of \( s \)-adic root valuation vectors in \( \text{Trop}_s(f) \cap \mathbb{Q}^n \) is much easier to prove than the opposite containment. Similarly, proving that \( \text{Amoeba}(f) \) is contained in a suitable neighborhood of \( \text{ArchTrop}(f) \) is easier than proving that \( \text{ArchTrop}(f) \) is contained in a suitable neighborhood of \( \text{Amoeba}(f) \): The first containment boils down to a careful application of geometric series, convex hulls, and the Triangle Inequality. The second containment involves a reduction to the univariate case and then an application of Rouché’s Theorem.

We close with a topological observation: We have seen that \( \text{ArchTrop}(f) \) need not be contained in \( \text{Amoeba}(f) \) nor have the same homotopy type as \( \text{Amoeba}(f) \), already for \( n = 1 \). (Consider, for instance, \( f(x_1) = (x_1 + 1)^2 \), which yields \( \text{ArchTrop}(f) = \{ \pm \log 2 \} \) but \( \text{Amoeba}(f) = \{ 0 \} \).) However, one can in fact always recover \( \text{ArchTrop}(f) \) as the limit of a sequence of suitably scaled amoebae. To clarify this, first recall that the Hausdorff distance between any two subsets \( X, Y \subseteq \mathbb{R}^n \) is

\[
\Delta(X, Y) := \max \left\{ \sup_{x \in X} \inf_{y \in Y} |x - y|, \sup_{y \in Y} \inf_{x \in X} |x - y| \right\}.
\]

Also, the support of a Laurent polynomial \( f(x) = \sum_{i=1}^t c_i x^{a_i} \) is \( \text{Supp}(f) := \{ a_i \mid c_i \neq 0 \} \).

Corollary 1.20. Let \( f \) be any \( n \)-variate \( t \)-nomial with \( t \geq 3 \) and support \( A \). Then:

(1) \( \Delta(\text{ArchTrop}(f), \text{Amoeba}(f)) < (\log 9)t - \log \frac{41}{2} < 2.2t - 3.7 \).

(2) There exists family of Laurent polynomials \( (f_\mu)_{\mu \geq 1} \) with \( \text{Supp}(f_\mu) = \text{Supp}(f) \) for all \( \mu \geq 1 \) and \( \Delta \left( \frac{1}{\mu} \text{Amoeba}(f_\mu), \text{ArchTrop}(f) \right) \to 0 \text{ as } \mu \to \infty \).

Assertion (2) thus shows, in a refined way, how \( \text{ArchTrop}(f) \) provides a dequantization for families of multivariate polynomials with fixed support, in the spirit of [Vir01]. A precursor to Assertion (2), involving the piece-wise linear structure of the intersection of \( \text{Amoeba}(f) \) with a large sphere, appears in [GKZ94, Prop. 1.9, pg. 197]. A non-Archimedean precursor to Assertion (2) appears in [Mik04, Cor. 6.4]. Thanks to Assertion (1), we can prove Assertion (2) in 3 lines.

Proof of Corollary 1.20: Assertion (1) of Corollary 1.20 follows immediately from Assertion (2) of Theorem 1.11. Let us write \( f(x) = \sum_{i=1}^t c_i x^{a_i} \), define \( f_\mu(x) := \sum_{i=1}^t c_i^\mu x^{a_i} \), and observe that \( f_1 = f \).

Since \( |c_i e^{\xi_1}v| \geq |c_i e^{\xi_2}v| \iff |c_i e^{\xi_2}v| \mu \geq |c_i e^{\xi_1}v| \mu \), we immediately obtain that \( \text{ArchTrop}(f_\mu) = \mu \text{ArchTrop}(f) \). So then \( \Delta(\text{Amoeba}(f_\mu), \text{Trop}(f)) = \mu \Delta \left( \frac{1}{\mu} \text{Amoeba}(f_\mu), \text{Trop}(f) \right) \) and thus

\[
\Delta \left( \frac{1}{\mu} \text{Amoeba}(f_\mu), \text{Trop}(f) \right) < \frac{(\log 9)t - \log \frac{41}{2}}{\mu} \text{ for all } \mu \geq 1. \]
2. Background on Univariate Bounds and the Complexity of Amoeba Membership

To prepare for the proof of our main metric results we will first review some classical root norm bounds in the univariate case to recast them in terms of ArchTrop($f$). We then prove a refinement of Theorem 1.6 (Corollary 2.3), Theorem 1.6 in full, Theorem 1.7, and conclude with a sketch the proof of our first result (Theorem 1.15) on the hardness of deciding point membership for Amoebae.

We begin with a pair of bounds dating back to the early 20th century, if not earlier.

Theorem 2.1.
(1) Suppose $f(x_1) = \sum_{i=0}^{d} c_i x_1^i \in \mathbb{C}[x_1]$ has a root $\zeta \in \mathbb{C}$ and $c_0 c_d \neq 0$. Then

$$\frac{1}{2} \min_{i \in \{1, \ldots, d\}} \left| \frac{c_i}{c_1} \right|^{1/i} < |\zeta| < 2 \max_{i \in \{0, \ldots, d-1\}} \left| \frac{c_i}{c_0} \right|^{1/(d-i)}.$$

(2) Suppose $f(x_1) := c_0 + \cdots + c_p x_1^p + \gamma_1 x_1^{n_1} + \cdots + \gamma_q x_1^{n_q} \in \mathbb{C}[x_1]$, $c_p \neq 0$, and $1 \leq p < n_1 < \cdots < n_q$.

Then $f$ has a root with absolute value

$$\left| \frac{c_0}{c_p} \right|^{1/p} \left( \frac{p+q}{q} \right)^{1/p}. \blacksquare$$

Bound (1) dates back to early 20th-century work of Fujiwara [Fuj16] (see also [RS02, pp. 243–249], particularly Bound 8.1.11 on pg. 247). Bound (2) was proved by Montel [Mon23] (see also [RS02, Thm. 9.5.1, pg. 304]). If one makes an elementary observation on the definition of ArchTrop($f$) then one immediately obtains a refinement of Theorem 1.6 for the roots of $f$ of largest and smallest (nonzero) norm. In what follows, a lower edge of a polygon in $\mathbb{R}^2$ is simply an edge possessing an outer normal of the form $(v, -1)$.

Proposition 2.2. For any univariate $t$-nomial with $t \geq 1$ we have that ArchTrop($f$) is the set of slopes of the lower edges of ArchNewt($f$). \blacksquare

Corollary 2.3. Suppose $f \in \mathbb{C}[x_1]$ is a univariate $t$-nomial with $t \geq 2$, degree $d$, and nonzero roots $\zeta_1, \ldots, \zeta_d$ (counting multiplicity) ordered so that $|\zeta_1| \leq \cdots \leq |\zeta_d|$. Then

(a) $- \log 2 < \log |\zeta_1| - \min \text{ArchTrop}(f) \leq \log(t - 1)$,

(b) $- \log(t - 1) \leq \log |\zeta_d| - \max \text{ArchTrop}(f) < \log 2$,

(c) The log 2 (resp. log(t−1)) terms above can not be replaced by any smaller constant (resp. function of $t$ solely).

Proof: The lower bound from Part (a) and the upper bound from Part (b) follow immediately from Proposition 2.2, upon taking the log absolute value of both sides of Bound (1) from Theorem 2.1. In particular, we see that the lower and upper bounds from Bound (1) are exactly $\frac{1}{2} e^{\min \text{ArchTrop}(f)}$ and $2 e^{\max \text{ArchTrop}(f)}$.

The upper bound from Part (a) follows similarly, but employing Bound (2) from Theorem 2.1 instead of Bound (1). In particular, one must apply Bound (2) in the following way: Take $p$ so that the $(p, - \log |c_p|)$ is the right-hand vertex of the left-most lower edge of ArchNewt($f$). By construction, this edge has slope $\frac{\log |c_0| - \log |c_p|}{p}$. Observing that $(\frac{p+q}{q})^{1/p} = \left( \frac{q+1}{p+1} \cdots (\frac{q+1}{p}) \right)^{1/p} \leq ((q+1)^{p})^{1/p} = q + 1$, and that the number of terms is $t = p + q + 1$ with $p \geq 1$, we are done.

The lower bound from Part (b) follows by applying the preceding paragraph to the polynomial $x_1^q f(1/x_1^q)$: This has the effect of reflecting ArchNewt($f$) across the vertical line $\frac{q}{2} \times \mathbb{R}$, and thus ArchTrop($f$) is replaced $-\text{ArchTrop}(f)$. So we ultimately prove an upper bound of $\log(t-1)$ on $-\log |\zeta_d| - (- \max \text{ArchTrop}(f))$ and we are done.
The optimality of the log 2 terms is evinced by the polynomials \( f_1(x_1) := x_1^{t-1} - x_1^{t-2} - \cdots - 1 \) and \( f_2(x_1) := -1 + x_1 + \cdots + x_1^{t-1} \). One need only show that \( f_1 \) (resp. \( f_2 \)) has a unique positive root increasing toward a limit of 2 (resp. decreasing toward a limit of \( \frac{1}{2} \)) as \( t \to \infty \). Uniqueness follows from Descartes’ Rule, while the limiting behavior of the positive root is easily obtained by Rolle’s Theorem and geometric series.

The optimality of the log \((t-1)\) terms is easily seen via the polynomial \( g(x_1) := (x_1 + 1)^{t-1} \). The left-most (resp. right-most) lower edge of \( \text{ArchNewt}(g) \) has slope \( -\log(t-1) \) (resp. \( \log(t-1) \)), by the log-concavity of the binomial coefficients. So by Proposition 2.2, \( \min \text{ArchTrop}(g) = -\log(t-1) \) and \( \max \text{ArchTrop}(g) = \log(t-1) \). Since Amoeba\((g) = \{0\} \), we are done.

We now recall a seminal collection of bounds due to Ostrowski:

**Theorem 2.4.** [Ost40a, Cor. IX, pg. 143]

Following the notation of Corollary 2.3, let \( d \) be the degree of \( f \), and let \( v_i \) denote the slope of the unique lower edge of the polygon \( \text{ArchNewt}(f) \cap ([i-1,i] \times \mathbb{R}) \). Then

\[
\begin{align*}
\text{(1)} & \quad -\log 2 < \log |\zeta_i| - v_i \leq \log d, \\
\text{(2)} & \quad -\log d \leq \log |\zeta_d| - v_d < \log 2, \\
\text{(3)} & \quad \log \left(1 - \frac{1}{2^{i+1}}\right) < \log |\zeta_i| - v_i < -\log \left(1 - \frac{1}{2^{i/(d-i+1)}}\right) \quad \text{for all } i \in \{2, \ldots, d-1\}.
\end{align*}
\]

In particular, \( -0.5348 \leq \log \left(1 - \frac{1}{2^{i+1}}\right) - (-\log i) < -0.3665 \) and

\[
0.3665 < -\log \left(1 - \frac{1}{2^{i/(d-i+1)}}\right) - \log(d-i+1) \leq 0.5348.
\]

**Remark 2.5.** Thanks to Proposition 2.2 we have \( \text{ArchTrop}(f) = \{v_1, \ldots, v_d\} \). In particular, note that our Theorem 1.6 implies that any given \( |\zeta_i| \) lies within distance \( \log 3 \) of some \( v_j \), possibly with \( j \neq i \). In this sense, the final assertion of Theorem 2.4 tells us that Theorem 1.6 isolates each \( |\zeta_i| \) strictly better than Ostrowski’s bounds, except possibly in the cases \( i \in \{2, d-1\} \) or \( t = d+1 = 3 \).

We have so far concentrated on showing that each log \( |\zeta_i| \) is close to some \( v_j \), with near-optimal distance bounds. Showing that each \( v_j \) is close to some \( \log |\zeta_i| \) appears to be more difficult, so we postpone the proof until Section 3.4.

### 2.1. Proving Theorem 1.6.

We will need two technical results, on bounding the norms of summands of sparse polynomials, before proving Theorem 1.6.

**Proposition 2.6.** Suppose \( f(x) := \sum_{j=1}^t c_j x^{a_j} \in \mathbb{C}[x^{\pm 1}] \) satisfies \( t \geq 3 \), \( a_1 < \cdots < a_t \), and \( c_j \neq 0 \) for all \( j \). Suppose further that \( w \in \text{ArchTrop}(f) \) and \( \ell \) is the unique index such that \((a_\ell, -\log |c_\ell|)\) is the right-hand vertex of the lower edge of \( \text{ArchNewt}(f) \) of slope \( w \) (so \( 2 \leq \ell \)).

Then for any \( N \in \mathbb{N} \) and \( x \) with \( |x| \geq (N+1)e^w \) we have \( \left| \sum_{j=1}^{\ell-1} c_j x^{a_j} \right| < \frac{1}{N} |c_\ell x^{a_\ell}|. \)

\footnote{There was a typo in Ostrowski’s original statement of the upper bound from Assertion (3), later corrected in an addendum by Ostrowski [Ost40b].}
\textbf{Proof:} First note that \(2 \leq \ell \leq t\) by construction. Letting \(r := \log |x|\) and \(\beta_j := \log |c_j|\) we obtain
\[
\left| \sum_{j=1}^{\ell-1} c_j x^{a_j} \right| \leq \sum_{j=1}^{\ell-1} |c_j x^{a_j}| = \sum_{j=1}^{\ell-1} e^{a_j r + \beta_j} = \sum_{j=1}^{\ell-1} e^{a_j (r-w) + a_j w + \beta_j}.
\]
Clearly, \(a_j \leq a_\ell - (\ell - j)\), so for \(r \geq w\) we have
\[
\left| \sum_{j=1}^{\ell-1} c_j x^{a_j} \right| \leq \sum_{j=1}^{\ell-1} e^{(a_\ell - (\ell - j))(r-w) + a_j w + \beta_j} \leq \sum_{j=1}^{\ell-1} e^{(a_\ell - (\ell - j))(r-w) + a_\ell w + \beta_\ell},
\]
where the last inequality follows from Proposition 2.2 and the definition of \(\text{ArchTrop}(f)\). So then
\[
\left| \sum_{j=1}^{\ell-1} c_j x^{a_j} \right| \leq e^{(a_\ell - (\ell - 1))(r-w) + a_\ell w + \beta_\ell} \sum_{j=1}^{\ell-1} e^{(j-1)(r-w)}
\]
\[
= e^{(a_\ell - (\ell - 1))(r-w) + a_\ell w + \beta_\ell} \left( \frac{e^{(\ell-1)(r-w)} - 1}{e^{r-w} - 1} \right)
\]
\[
< e^{(a_\ell - (\ell - 1))(r-w) + a_\ell w + \beta_\ell} \left( \frac{e^{(\ell-1)(r-w)} - 1}{e^{r-w} - 1} \right) = \frac{e^{a_\ell r + \beta_\ell}}{e^{r-w} - 1}
\]
So to prove our desired inequality it clearly suffices to enforce \(e^{r-w} - 1 \geq N\). The last inequality clearly holds for all \(r \geq w + \log(N+1)\), so we are done. \(\blacksquare\)

A simple consequence of our preceding term domination trick is that we can give explicit annuli in \(\mathbb{C}\) free of roots of \(g\).

\textbf{Corollary 2.7.} Suppose \(f(x) := \sum_{j=1}^{t} c_j x^{a_j} \in \mathbb{C}[x^{\pm 1}]\) satisfies \(a_1 < \cdots < a_t, c_j \neq 0\) for all \(j\), and that \(w_1\) and \(w_2\) are consecutive points of \(\text{ArchTrop}(f)\) satisfying \(w_2 \geq w_1 + \log 9\). Let \(\ell\) be the unique index such that \((a_\ell, - \log |c_\ell|)\) is the unique vertex of \(\text{ArchNewt}(f)\) incident to lower edges of slopes \(w_1\) and \(w_2\) (so \(2 \leq \ell \leq t-1\)). Then \(f\) has no root \(\zeta \in \mathbb{C}\) satisfying \(3e^{w_1} \leq |\zeta| \leq \frac{3}{2} e^{w_2}\).

\textbf{Proof:} Let \(A\) denote the stated annulus. By Proposition 2.6, we have \(\sum_{j=1}^{\ell-1} c_j \zeta^{a_j} \leq \frac{1}{2} |c_\ell \zeta^{a_\ell}|\) provided \(|\zeta| \geq 3e^{w_1}\). Employing the substitution \(x \mapsto \frac{1}{x}\) (which has the effect of replacing \(\text{ArchTrop}(f)\) by \(-\text{ArchTrop}(f)\)) we also obtain \(\left| \sum_{j=\ell+1}^{t} c_j \zeta^{a_j} \right| < \frac{1}{2} |c_\ell \zeta^{a_\ell}|\) provided \(\frac{1}{3} \geq 3e^{-w_2}\).

So we obtain \(\left| \sum_{j \neq \ell} c_j \zeta^{a_j} \right| < |c_\ell \zeta^{a_\ell}|\) in \(A\), and this inequality clearly contradicts the existence of a root of \(f\) in \(A\). \(\blacksquare\)

We use \#\(S\) to denote the cardinality of a set \(S\).

\textbf{Proof of Theorem 1.6:} That \(\#\text{ArchTrop}(f) \leq t - 1\) follows easily from Proposition 2.2: \(\#\text{ArchTrop}(f)\) is the number of lower edges of \(\text{ArchNewt}(f)\) and \(\text{ArchNewt}(f)\) has at most \(t\) vertices.

Now, if \(\#\text{ArchTrop}(f) = 1\) then
\[
\text{Amoeba}(f) \subset (\text{ArchTrop}(f) - \log 2, \text{ArchTrop}(f) + \log 2),
\]
regardless of \(t\), thanks to Corollary 2.3.

If \(\#\text{ArchTrop}(f) = 2\) and \(t = 3\), then Proposition 2.6 applied to \(f(x)\) and \(f(1/x)\) with \(N = 1\) implies that any point of \(\text{Amoeba}(f)\) must lie either in the open neighborhood \((\min \text{ArchTrop}(f))_{\log 2}\) or the open neighborhood \((\max \text{ArchTrop}(f))_{\log 2}\). Since \(1 \leq \#\text{ArchTrop}(f) \leq 2\) when \(t = 3\), this completes the \(t = 3\) case.
So let us now assume \( t \geq 4 \). It then suffices to prove that Amoeba(\( f \)) has no points at distance \( \log 3 \) (or greater) from every point of ArchTrop(\( f \)). This is implied by \( f \) not vanishing at any point with norm \( e^w \) with \( |w - v| \geq \log 3 \) for all \( v \in \text{ArchTrop}(f) \). But the latter assertion follows immediately from Corollary 2.7. ■

2.2. Proving Theorem 1.7. We are working toward proving that the open annuli in \( \mathbb{C} \) containing the roots of a univariate sparse polynomial \( f \) correspond exactly to clusters of “closely spaced” consecutive points of ArchTrop(\( f \)). A key step will be to relate clusters of points in ArchTrop(\( f \)) to certain sub-summands of \( f \). In particular, sets of consecutive “large” (resp. “small”) points of ArchTrop(\( f \)) will correspond to sums of “high” (resp. “low”) order terms of \( f \). So, to relate the roots of high (or low) order summands of \( f \) to an explicit portion of the roots of \( f \), let us first recall the following classical result.

Rouché’s Theorem. (See, e.g., [Ahl79].) Suppose \( U \subset \mathbb{C} \) is a simply connected open set with compact closure \( \bar{U} \). Let \( g_1 \) and \( g_2 \) be functions meromorphic on \( \bar{U} \) with only finitely many zeroes and poles in \( \bar{U} \), no removable singularities in \( \bar{U} \), and no zeroes or poles on the boundary \( \partial U \). Assume also that \( |g_1| < |g_2| \) on \( \partial U \). Then \( g_1 + g_2 \) and \( g_2 \) have the same number of roots, counting multiplicities, in \( U \). (We count poles as zeroes with negative multiplicity.) ■

Lemma 2.8. Let \( f(x) := \sum_{j=1}^{t} c_j x^{a_j} \) with \( a_1 < \cdots < a_t \) and \( c_j \neq 0 \) for all \( j \), and set \( w_{\text{min}} := \min \text{ArchTrop}(f) \) and \( w_{\text{max}} := \max \text{ArchTrop}(f) \). Also let \( w_1 \) and \( w_2 \) be consecutive points of ArchTrop(\( f \)) satisfying \( w_2 - w_1 \geq \log 9 \), and let \( \ell \) be the unique index such that \( (a_{\ell}, -\log |c_{\ell}|) \) is the unique vertex of ArchNewt(\( f \)) incident to lower edges of slopes \( w_1 \) and \( w_2 \) (so \( 2 \leq \ell \leq t - 1 \)). Then, counting multiplicities, \( f \) has exactly \( a_{\ell} - a_1 \) (resp. \( a_1 - a_{\ell} \)) roots \( \zeta \in \mathbb{C} \) satisfying \( \frac{1}{2} e^{w_{\text{min}}} < |\zeta| < 3 e^{w_1} \) (resp. \( \frac{1}{3} e^{w_2} < |\zeta| < 2 e^{w_{\text{max}}} \)).

In what follows, we let \( D(r) \) (resp. \( \bar{D}(r) \)) denote the open (resp. closed) disk of radius \( r \), centered at the origin, in \( \mathbb{C} \).

Proof of Lemma 2.8: By symmetry (with respect to replacing \( x \) by \( \frac{1}{x} \)) it clearly suffices to prove the first root count. Proposition 2.6 then tells us that \( \frac{1}{2} |c_{\ell} \zeta^{a_\ell}| > \left| \sum_{j=1}^{\ell-1} c_j \zeta^{a_j} \right| \) and, by another application of Proposition 2.6 to \( f(1/x) \) (remembering that \( w_2 - w_1 \geq \log 9 \)), we also obtain \( \frac{1}{2} |c_{\ell} \zeta^{a_\ell}| > \left| \sum_{j=\ell+1}^{t} c_j \zeta^{a_j} \right| \) when \( |\zeta| = 3 e^{w_1} \). So \( |c_{\ell} \zeta^{a_\ell}| > \left| \sum_{j \neq \ell} c_j \zeta^{a_j} \right| \) when \( |\zeta| = 3 e^{w_1} \). So by the Rouché’s Theorem we must have that the monomial \( c_{\ell} x^{a_\ell} \) and \( f \) have the same total number of roots and poles, counting multiplicities (poles counted with negative multiplicity), in the open disk \( D(3e^{w_1}) \). Since \( f \) has no roots in the closed disk \( \bar{D} \left( \frac{1}{2} e^{w_{\text{min}}} \right) \), other than a root/pole of multiplicity \( a_1 \) at the origin, we obtain that \( f \) has exactly \( a_{\ell} - a_1 \) roots \( \zeta \in \mathbb{C} \) (and no poles), counting multiplicity, satisfying \( \frac{1}{2} e^{w_{\text{min}}} < |\zeta| < 3 e^{w_1} \). ■

Proof of Theorem 1.7: First note that, by definition, \( \Gamma \) must contain at least 1 point of ArchTrop(\( f \)) and thus \( \Lambda_{\Gamma} \) is a positive integer.

Now suppose ArchTrop(\( f \))_{\log 3} is connected. Then \( \Lambda_{\Gamma} = a_{\ell} - a_1 \), Corollary 2.3 tells us that all the roots of \( f \) in fact lie in \( \Gamma \), and we are done.

So assume ArchTrop(\( f \))_{\log 3} has at least 2 distinct connected components. Lemma 2.8 then immediately yields the conclusion of Theorem 1.7 when \( \Gamma \) is either the left-most or right-most connected component of ArchTrop(\( f \))_{\log 3}: We simply take \( w_1 \) to be the right-most point of
\( \Gamma \cap \text{ArchTrop}(f) \) and \( w_2 \) the left-most point of \( \text{ArchTrop}(f) \) in the connected component of \( \text{ArchTrop}(f)_{\log 3} \) immediately to the right of \( \Gamma \), or we take \( w_2 \) to be the left-most point of \( \Gamma \cap \text{ArchTrop}(f) \) and \( w_1 \) the right-most point of \( \text{ArchTrop}(f) \) in the connected component of \( \text{ArchTrop}(f)_{\log 3} \) immediately to the left of \( \Gamma \).

Noting that

\[
\sum_{\Gamma \text{ a connected component of } \text{ArchTrop}(f)_{\log 3}} \Lambda_\Gamma = a_t - a_1,
\]

we can now proceed by induction on the number of connected components of \( \text{ArchTrop}(f)_{\log 3} \): We simply ignore the left-most and right-most connected components of \( \text{ArchTrop}(f)_{\log 3} \), and treat the new left-most and right-most connected components via Lemma 2.8 as in the last paragraph. So we are done. ■

2.3. Classical Computational Algebra and Amoeba Membership. Let us first recall the following results of Plaisted and Chistov and Grigoriev.

**Theorem 2.9.** [Pla84] The following problem:

- Decide whether an arbitrary input \( f \in \mathbb{Z}[x_1] \) has a complex root of norm 1.

is NP-hard. ■

**Theorem 2.10.** [CG84, Can88] There is an algorithm that, given any collection of polynomials \( f_1, \ldots, f_p, g_1, \ldots, g_q, h_1, \ldots, h_r \in \mathbb{Q}[x_1, \ldots, x_n] \), decides whether there is a \( \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n \) with \( f_1(\zeta) = \cdots = f_p(\zeta) = 0, g_1(\zeta), \ldots, g_q(\zeta) > 0, \) and \( h_1(\zeta), \ldots, h_r(\zeta) \geq 0 \), in time

\[
O\left( \sum_{i=1}^p \text{size}(f_i) \right) + O\left( \sum_{i=1}^q \text{size}(g_i) \right) + O\left( \sum_{i=1}^r \text{size}(h_i) \right) \circ (1),
\]

using \( \sum_{i=1}^p \text{size}(f_i) \) processors. ■

**Sketch of Proof of Theorem 1.15:** First observe that \( \text{Log}|v| \in \text{Amoeba}(f) \iff f \) has a complex root \( \zeta \) with \( |\zeta| = |v| \). Letting \( A \) and \( B \) denote the real and imaginary parts of \( f \), and \( \alpha_i \) and \( \beta_i \) denote the real and imaginary parts of \( \zeta_i \), we thus obtain that \( \text{Log}|v| \in \text{Amoeba}(f) \) if and only if there are \( \alpha, \beta \in \mathbb{R}^n \) with

\[
A(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n) = 0
\]

\[
B(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n) = 0
\]

\[
\alpha_1^2 + \beta_1^2 = |v_1|^2
\]

\[
\vdots
\]

\[
\alpha_n^2 + \beta_n^2 = |v_n|^2
\]

for some \( \alpha, \beta \in \mathbb{R}^n \). Now, while the preceding system of equations has size significantly larger than \( \text{size}(v) + \text{size}(f) \) (due to the underlying expansions of powers of \( x_i = \alpha_i + \sqrt{-1} \beta_i \)), we can introduce new variables to obtain another polynomial system, also with a real solution if and only if \( \text{Log}|v| \in \text{Amoeba}(f) \), with size linear in \( \text{size}(v) + \text{size}(f) \) instead. Applying Theorem 2.10, we obtain our \( \text{PSPACE} \) upper bound.

Our NP-hardness complexity lower bound follows immediately from Theorem 2.9, since \( |\zeta| = 1 \implies \text{Log}|\zeta| = 0 \). ■

**Remark 2.11.** Theobald makes substantially the same observations in [The02, Sec. 2.2], but with a looser notion of input size, and omitting speed-ups via parallelizing. In particular, in our notation, his complexity upper bound would be \( \text{EXPTIME} \). ✡
3. The Proof of Theorem 1.11

The assertion that \( t \geq k + 1 \) follows immediately since any \( k \)-dimensional polytope always has at least \( k + 1 \) vertices, and \( \text{Newt}(f) \) has at most \( t \) vertices.

3.1. Proof of Assertion (0). Note that, by definition, \( \text{ArchTrop}(f) \) is a section of the normal fan \( \mathcal{F} \) of \( \text{ArchNewt}(f) \). Each \((n-1)\)-cell of \( \text{ArchTrop}(f) \) is the section of a unique \( n \)-cell of \( \mathcal{F} \) dual to an edge of \( \text{ArchNewt}(f) \). Since \( \text{ArchNewt}(f) \) has at most \( t \) vertices, it has at most \( \binom{t}{2} \) edges and we obtain our upper bound.

Let us call any face of \( \text{ArchNewt}(f) \) possessing an outer normal of the form \((v,-1)\) a lower face. Note that any path between points in \( \text{ArchTrop}(f) \) induces a (connected) sequence of faces of \( \text{ArchTrop}(f) \). So by duality again, \( \text{ArchTrop}(f) \) is connected if and only if \( L \setminus L_0 \) is path-connected, where \( L \) (resp. \( L_0 \)) is the union of all lower faces (resp. lower vertices) of \( \text{ArchNewt}(f) \). The set \( L \setminus L_0 \) is topologically a \( k \)-ball minus a finite collection of points, and is thus path-connected for \( k \geq 2 \). In particular, \( k=1 \) implies that \( \text{ArchNewt}(f) \) lies in a 2-plane and has at most \( t-1 \) lower edges. So we are done. ■

3.2. Proof of Assertion (1). First note that the definitions of \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \) are invariant under translation of \( \{a_1, \ldots, a_{k+1}\} \). So we may assume without loss of generality that \( a_1 \) is the origin \( O \). Furthermore, since \( k = \dim \text{Conv}(\{a_1, \ldots, a_{k+1}\}) \), we have that \( a_2, \ldots, a_{k+1} \) are linearly independent in \( \mathbb{Q}^n \).

Defining the vector of monomials \( x^B := \left(x_1^{b_{11}} \cdots x_n^{b_{n1}}, \ldots, x_1^{b_{1n}} \cdots x_n^{b_{nn}}\right) \) for any \( n \times n \) matrix \( B \) with \((i,j)\)-entry \( b_{ij} \in \mathbb{Q} \), it is easily checked that the map \( m(x) = x^B \) is an analytic automorphism of \( \mathbb{C}^n \setminus \{0\} \) when \( B \) is invertible. In particular, such a map induces the linear map \( x \mapsto B^{-1}x \) on both \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \). Since invertible real linear maps are homeomorphisms, they preserve containment and contractability for \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \). Letting \( B \) be the matrix whose inverse has columns \( a_2, \ldots, a_{k+1}, b_1, \ldots, b_{n-k} \) (for any \( b_1, \ldots, b_{n-k} \in \mathbb{Q} \) with \( \{a_2, \ldots, a_{k+1}, b_1, \ldots, b_{n-k}\} \) forming a basis for \( \mathbb{Q}^n \)), we may thus restrict our study of \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \) to the special case \( k = n \) and \( f(x) = c_0 + c_1 x_1 + \cdots + c_n x_n \). Moreover, since rescaling \( x \) merely affects both \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \) by a common translation, we may further restrict to the special case \( f(x) = 1 + x_1 + \cdots + x_n \).

It is then easily checked that \( \text{ArchTrop}(f) \) is simply the normal fan of the standard \( n \)-simplex \( \Delta_n := \text{Conv}(\{O, e_1, \ldots, e_n\}) \) in \( \mathbb{R}^n \). Note in particular that \( \text{ArchTrop}(f) \) is thus contractible, and its complement consists of \( n+1 \) open cones, each with boundary combinatorially equivalent to the boundary of the negative orthant in \( \mathbb{R}^n \). Note also that we can map any vertex of \( \Delta_n \) to any other vertex of \( \Delta_n \) via an invertible affine map, and this affine map also preserves containment and contractability for \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \).

So to prove containment, it suffices to prove (i) \( \text{Amoeba}(1 + x_1 + \cdots + x_n) \) contains the boundary of the negative orthant; and to prove contractibility, it suffices to prove (ii) \( \text{ArchTrop}(1 + x_1 + \cdots + x_n) \) is a strong deformation retract of \( \text{Amoeba}(f) \). In particular, since Example 1.4 already includes the case \( n = 1 \), we may assume henceforth that \( n \geq 2 \).

To prove (i), let \( v \) be any point of the boundary of the negative orthant in \( \mathbb{R}^n \). By symmetry, we may assume that \( v_{j+1} = \cdots = v_n = 0 \), and \( v_1, \ldots, v_j < 0 \) for some \( j \in [n-1] \). Observe then that we can always find \( \theta_1, \ldots, \theta_j \in [0, 2\pi) \) so that

\[
\left| e^{v_1} e^{\theta_1} + \cdots + e^{v_j} e^{\theta_j} \right| \leq \max_{i \in [j]} \{e^{v_i}\},
\]
and the phase of $e^{v_1\theta_1 \sqrt{-1}} + \cdots + e^{v_\tau \theta_\tau \sqrt{-1}}$ takes any prescribed value. (The case $j = 1$ is trivial, while the remaining cases follows easily by induction, using the Triangle Inequality.) Similarly, we may pick $\theta_{j+1}, \ldots, \theta_n \in [0, 2\pi)$ so that $1 + e^{\sqrt{-1} \theta_{j+1}} + \cdots + e^{\sqrt{-1} \theta_n}$ attains any fixed value in the unit circle centered at 1 (resp. the closed unit disk centered at 1) when $j = n - 1$ (resp. $j < n - 1$). In particular, we may pick $\theta_1, \ldots, \theta_n$ so that $1 + e^{\sqrt{-1} \theta_1} + \cdots + e^{\sqrt{-1} \theta_n} = 0$. Setting $x_i = e^{\sqrt{-1} \theta_i}$, this proves (i), and thus $\text{ArchTrop}(f) \subset \text{Amoeba}(f)$.

By then considering the distance of each point of the boundary of $\text{Amoeba}(f)$ to the nearest top-dimensional cell of $\text{ArchTrop}(f)$, we can easily construct a strong deformation retraction from $\text{Amoeba}(f)$ to $\text{ArchTrop}(f)$. In particular, since we’ve assumed $n \geq 2$, we have $t \geq 3$, and thus $\text{Amoeba}(f) \subset \text{ArchTrop}(f)^{\log 3}$ by Assertion (2a) (proved independently in the next section). Since we already have $\text{ArchTrop}(f) \subset \text{Amoeba}(f)$, we then merely employ a deformation of $\mathbb{R}^n$ that retracts an open $(\log 3)$-neighborhood of the fan $\text{ArchTrop}(f)$ to $\text{ArchTrop}(f)$. \hfill \blacksquare

3.3. Proof of Part (a) of Assertion (2). Let $w := (\log |\zeta_1|, \ldots, \log |\zeta_n|) \in \text{Amoeba}(f)$ and assume without loss of generality that $|c_1 \zeta_1| \geq |c_2 \zeta_2| \geq \cdots \geq |c_t \zeta_t|$. Since $f(\zeta) = 0$ implies that $|c_1 \zeta_1| = |c_2 \zeta_2 + \cdots + c_t \zeta_t|$, the Triangle Inequality immediately implies that $|c_1 \zeta_1| \leq (t - 1)|c_2 \zeta_2|$. Taking logarithms, we then obtain

\begin{equation}
|a_1| \cdot w + \log |c_1| \geq \cdots \geq |a_t| \cdot w + \log |c_t| \quad \text{and}
\end{equation}

\begin{equation}
|a_1| \cdot w + \log |c_1| \leq \log(t - 1) + |a_2| \cdot w + \log |c_2|
\end{equation}

For each $i \in \{2, \ldots, t\}$ let us then define $\delta_i$ to be the shortest vector such that $a_1 \cdot (w + \delta_i) + \log |c_1| = a_i \cdot (w + \delta_i) + \log |c_i|$. Note that $\delta_i = \lambda_i \cdot (a_i - a_1)$ for some nonnegative $\lambda_i$ since we are trying to affect the dot-product $\delta_i \cdot (a_1 - a_i)$. In particular, $\lambda_i = \frac{(a_1 - a_i) \cdot w + \log |c_1|/c_1}{\log(t - 1) \cdot |a_1 - a_i|^2}$ so that $|\delta_i| = \frac{(a_1 - a_i) \cdot w + \log |c_1|/c_1}{\log(t - 1) \cdot |a_1 - a_i|^2}$. (Indeed, Inequality (1) implies that $(a_1 - a_i) \cdot w + \log |c_1|/c_1 \geq 0$.)

Inequality (2) implies that $(a_1 - a_2) \cdot w + \log |c_1|/c_2 \leq \log(t - 1)$. We thus obtain $|\delta_2| \leq \frac{\log(t - 1)}{\log |a_1 - a_2|^2} \leq \log(t - 1)$. So let $i_0 \in \{2, \ldots, t\}$ be any $i$ minimizing $|\delta_i|$. We of course have $|\delta_{i_0}| \leq \log(t - 1)$, and by the definition of $\delta_{i_0}$ we have $a_1 \cdot (w + \delta_{i_0}) + \log |c_1| = a_{i_0} \cdot (w + \delta_{i_0}) + \log |c_{i_0}|$.

Moreover, the fact that $\delta_{i_0}$ is the shortest among the $\delta_i$ implies that $a_1 \cdot (w + \delta_{i_0}) + \log |c_1| \geq a_i \cdot (w + \delta_{i_0}) + \log |c_i|$ for all $i$. Otherwise, we would have $a_1 \cdot (w + \delta_{i_0}) + \log |c_1| < a_i \cdot (w + \delta_{i_0}) + \log |c_i|$ and $a_1 \cdot w + \log |c_1| \geq a_i \cdot w + \log |c_i|$ (the latter following from Inequality (1)). Taking a convex linear combination of the last two inequalities, it is then clear that there must be a $\mu \in [0, 1)$ such that $a_1 \cdot (w + \mu \delta_{i_0}) + \log |c_1| = a_i \cdot (w + \mu \delta_{i_0}) + \log |c_i|$. Thus, by the definition of $\delta_i$, we would obtain $|\delta_i| \leq \mu \cdot |\delta_{i_0}| < |\delta_{i_0}|$ — a contradiction.

We thus have the following:

\begin{align*}
a_1 \cdot (w + \delta_{i_0}) - (-\log |c_1|) &= a_{i_0} \cdot (w + \delta_{i_0}) - (-\log |c_{i_0}|), \\
a_1 \cdot (w + \delta_{i_0}) - (-\log |c_1|) &\geq a_i \cdot (w + \delta_{i_0}) - (-\log |c_i|)
\end{align*}

for all $i$, and $|\delta_{i_0}| \leq \log(t - 1)$. This implies that $w + \delta_{i_0} \in \text{ArchTrop}(f)$. In other words, we’ve found a point in $\text{ArchTrop}(f)$ sufficiently near $\log |\zeta|$ to prove our desired upper bound. \hfill \blacksquare

3.4. Proving Part (b) of Assertion (2). We begin with the special case $n = 1$.

**Theorem 3.1.** Suppose $f$ is any univariate $t$-nomial with $t \geq 3$ and $s := \# \text{ArchTrop}(f)$. (So $1 \leq s \leq t - 1$.) Then for any $v \in \text{ArchTrop}(f)$ there is a root $\zeta \in \mathbb{C}$ of $f$ with $|v - \log |\zeta|| < \log 2$, \hfill \blacksquare
\[ |v - \log |\zeta|| \leq \log \min \{18, t - 1\}, \text{ or } |v - \log |\zeta|| < (\log 9)s - \log \frac{9}{2} < 2.2s - 1.5, \text{ according as } s \text{ is } 1, 2, \text{ or } \geq 2. \]

**Proof:** Let \( \Gamma \) be the connected component of \( \text{ArchTrop}(f)_{\log 3} \) containing \( v \in \text{ArchTrop}(f) \) and \( m := \# (\Gamma \cap \text{ArchTrop}(f)) \). (So \( 1 \leq m \leq s \).) By Theorem 1.7, there is a root \( \zeta \in \mathbb{C} \) of \( f \) with \( \log |\zeta| \in \Gamma \). By Corollary 2.3, we must also have \[ \min \text{ArchTrop}(f) - \log 2 < \log |\zeta| < \max \text{ArchTrop}(f) + \log 2. \]

The quantity \( |v - \log |\zeta|| \) is thus clearly maximized, for instance, when \( v \) is as far to the left as possible and \( \log |\zeta| \) is as far to the right as possible. In other words, \[ |v - \log |\zeta|| < \log(3) + (\log 9)(m - 2) + \log(3) + \delta, \]

where \( \delta \) is \( \log 3 \) or \( \log 2 \), according as \( m < s \) or \( m = s \). We thus obtain the largest possible upper bound of \( (\log 9)s - \log \frac{9}{2} \) when \( m = s \). So now we merely need to refine the cases with \( s \in \{1, 2\} \).

The case \( s = 1 \) follows immediately from Corollary 2.3 since \( \min \text{ArchTrop}(f) = \max \text{ArchTrop}(f) \) here.

The case \( s = 2 \) proceeds as follows: If \( m = 1 \) then \( \Gamma \) is an open interval of width \( 2 \log 3 \) with \( v \) at its median, so we must have \( |v - \log |\zeta|| < \log 3 \). If \( m = 2 \) then \( \Gamma \) is an open interval of width at most \( 4 \log 3 \), but we still have \[ \min \text{ArchTrop}(f) - \log 2 < \log |\zeta| < \max \text{ArchTrop}(f) + \log 2. \]

So then, \( |v - \log |\zeta|| \) can again be maximized by having \( v \) as far left as possible and \( \log |\zeta| \) as far as possible. In particular, \( s = 2 \) implies that \( \text{ArchTrop}(f) = \{\min \text{ArchTrop}(f), \max \text{ArchTrop}(f)\} \).

So we obtain \( |v - \log |\zeta|| < \log(3) + \log(3) + \log(2) = \log 18 \). In addition, we can apply Corollary 2.3 to observe that the smallest value of \( \min \text{ArchTrop}(f) - \log |\zeta| \) (over all roots \( \zeta \in \mathbb{C} \setminus \{0\} \)) can be no larger than \( \log(t - 1) \). So we obtain \( |v - \log |\zeta|| \leq \log \min \{18, t - 1\} \).

**Proof of Part (b) of Assertion (2):** Let \( v \in \text{ArchTrop}(f) \) be arbitrary. We will first reduce to the case \( n = 1 \) by finding a \( u \in \text{Amoeba}(f) \) agreeing with \( v \) in the last \( n - 1 \) coordinates, and with the first coordinates of \( u \) and \( v \) sufficiently close.

If \( v \in \text{Amoeba}(f) \) then there is nothing to prove. So let us assume \( v \notin \text{Amoeba}(f) \). Let us specialize \( \zeta_2, \ldots, \zeta_n \) so that \( \log |\zeta_i| = v_i \) for all \( i \in \{2, \ldots, n\} \). (Note that we may choose the phases of \( \zeta_2, \ldots, \zeta_n \) freely.) If we can find a \( u_1 \) with \( f(x_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}[x_1] \) having a root of norm \( u_1 \) satisfying \( |u_1 - v_1| < (\log 9)t - \log \frac{81}{2} \) then we will clearly be done. Since, after specializing \( x_2, \ldots, x_n \), the number of terms of \( f \) can not increase, we can thus assume that \( n = 1 \). By Theorem 3.1, we are done.

**3.5. Proof of Assertion (3).** To prove Part (a), note that \( (\frac{-1}{\epsilon - 1}, \ldots, \frac{-1}{\epsilon - 1}) \) is a root of \( \varphi \) and thus \( p := (\log |\frac{-1}{\epsilon - 1}|, \ldots, \log |\frac{-1}{\epsilon - 1}|) \in \text{Amoeba}(\varphi) \). Also, from our proof of Assertion (1), we know that the intersection of the negative orthant in \( \mathbb{R}^{t-1} \) with \( \text{ArchTrop}(\varphi) \) is the boundary of the negative orthant. So the distance from \( p \) to \( \text{ArchTrop}(\varphi) \) is \( \log(t - 1) \), and \( \text{ArchTrop}(\varphi) \) can contain \( p \) only for \( \epsilon \geq \log(t - 1) \).

To prove Part (b), note that \( (x_1 + 1)^{t-k} \) has a unique root of multiplicity \( t - k \) at \( x_1 = -1 \). Recall that the roots of a monic univariate polynomial are continuous functions of the coefficients, e.g., [RS02, Thm. 1.3.1, pg. 10].\(^2\) Clearly then, for \( |x_2|, \ldots, |x_n| \) sufficiently small (or \( z_2 := \log |x_2|, \ldots, z_n := \log |x_n| \) sufficiently negative), the distance from \( \text{Amoeba}(f) \) to the hyperplane \( \{0\} \times \mathbb{R}^{n-1} \) can be made arbitrarily small.

\(^2\)The statement there excludes roots of multiplicity equal to the degree of the polynomial but the proof in fact works in this case as well.
On the other hand, since the slopes of the lower edges of ArchNewt\((x_1 + 1)^{t-k}\) are exactly \(\{-\log \left( \binom{t-k}{i} / \binom{t-1}{i} \right) \}_{i \in [t-k]} = \{\log \left( \frac{x}{t-k-i+1} \right) \}_{i \in [t-k]}\), this will enable us to prove that ArchTrop\((f)\) contains a ray of the form \(\{(\log(t-k), N, \ldots, N)\}_{N \to +\infty}\) and thus conclude: The points along the ray have distance to Amoeba\((f)\) approaching \(\log(t-k)\), by our last paragraph.

To see why such a ray lies in ArchTrop\((x_1 + 1)^{t-k}\) simply note that as \(N \to -\infty\), the linear form \(\log(t-k)u_1 + Nu_2 + \cdots + Nu_n - u_{n+1}\) is maximized exactly at the vertices \((t-k-1, 0, \ldots, 0, -\log(t-k))\) and \((t-k, 0, \ldots, 0, 0)\) of ArchNewt\((x_1 + 1)^{t-k}\). Indeed, the only other possible vertices of ArchNewt\((x_1 + 1)^{t-k}\) are the basis vectors \(e_2, \ldots, e_k\) of \(\mathbb{R}^{n+1}\). So we are done.

4. Proving Theorem 1.17

Let us first recall the following result on comparing monomials in rational numbers.

**Theorem 4.1.** [BRS09, Sec. 2.4] Suppose \(\alpha_1, \ldots, \alpha_n \in \mathbb{Q}\) are positive and \(\beta_1, \ldots, \beta_n \in \mathbb{Z}\). Also let \(A\) be the maximum of the numerators and denominators of the \(\alpha_i\) (when written in lowest terms) and \(B := \max_i \{\|\beta_i\|\}\). Then, within

\[
O(n^30^n \log(B) (\log \log B)^2 \log \log \log(B) (\log \log A) (\log \log \log A)^n)
\]

bit operations, we can determine the sign of \(\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} - 1\).

While the underlying algorithm is a simple application of Arithmetic-Geometric Mean Iteration (see, e.g., [Ber03]), its complexity bound hinges on a deep estimate of Nesterenko [Nes03], which in turn refines seminal work of Matveev [Mat00] and Alan Baker [Bak77] on linear forms in logarithms.

**Proof of Theorem 1.17:** From Proposition 1.18, it is clear that we merely need an efficient method to compare quantities of the form \(|c_i t^{\alpha_i}|\), and there are exactly \(t-1\) such comparisons to be done. So our first complexity bound follows immediately from the case of Theorem 4.1 where \(A \leq 2^s\) and \(\beta_i \leq d\) for all \(i \in [2n+2]\). In particular, \(30^2 \log 2 < 623.8325\) and \(\sqrt{623.8325} < 25.2\).

The second assertion follows almost trivially: thanks to the exponential form of the coefficients and the query point, one can take logarithms of both sides to reduce to linear combinations involving integers of bit size max\(\{s, \log d\}\). So the signs can be decided by standard techniques for fast integer multiplication (see, e.g., [BS96, pg. 43]).

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Department of Mathematics, University of Zaragoza, Zaragoza, Spain.
E-mail address: mavendar@yahoo.com.ar

Department of Mathematics, Texas A&M University TAMU 3368, College Station, Texas 77843-3368, USA.
E-mail address: romwell@math.tamu.edu

Department of Mathematics, Texas A&M University TAMU 3368, College Station, Texas 77843-3368, USA.
E-mail address: nisse@math.tamu.edu

Department of Mathematics, Texas A&M University TAMU 3368, College Station, Texas 77843-3368, USA.
E-mail address: rojas@math.tamu.edu