Solutions to Selected Problems on HW#1

Note: In pretty much any mathematical problem beyond the level of high school algebra, there will be many possible solutions and many ways to elegantly write up the solution. So while your solution need not look exactly like what’s here, it should be direct and clearly written.

32 (pg. 19):
Solution #1: By brute-force, one can simply check that \( x = 4 \) is the only solution. There is nothing wrong with brute-force if it’s fast...
Solution #2: Observing that \( +_7 \) is the same as addition in \( \mathbb{Z}/7\mathbb{Z} \), we simply observe that \( x +_7 x +_7 x = 3x \mod 7 \). So we can then solve \( 3x = 5 \mod 7 \) instead. The latter can be done by brute-force, or using the Extended Euclidean Algorithm to find a pair of integers \( (x, y) \) with \( 3x + 7y = 5 \). The latter approach easily leads to \( 3 \cdot 4 + 7 \cdot (-1) = 5 \), and thus \( x = 4 \).

19 (pg. 26):
The function \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(a, b) := a - b \) is clearly well-defined, since the difference of any two real numbers is a real number. So \( \ast \) is indeed a binary operation.

23 (pg. 27):
(a) Yes: Letting \( \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix} \in H \) for \( i \in \{1, 2\} \), we easily obtain
\[
\begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{bmatrix}
\]
and the right-hand matrix is clearly in \( H \).
(b) Yes: Following the notation of Part (a), observe that
\[
\begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 - b_1b_2 & -a_1b_2 - b_1a_2 \\ b_1a_2 + a_1b_2 & -b_1a_2 + b_1a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 - b_1b_2 & -(a_1b_2 + a_2b_1) \\ b_1a_2 + a_1b_2 & a_1a_2 - b_1b_2 \end{bmatrix}
\]
and the right-hand matrix is clearly in \( H \).

8 (pg. 45):
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Additional Comments: This group is usually referred to as \( (\mathbb{Z}/8\mathbb{Z})^* \) — the group of multiplicative units of \( \mathbb{Z}/8\mathbb{Z} \). It is worth noting that the multiplication table above is, up to relabelling, the same as that of \( V \) (the Klein 4-group). In other words, \( (\mathbb{Z}/8\mathbb{Z})^* \) and \( V \) are isomorphic. Note also that \( (\mathbb{Z}/8\mathbb{Z})^* \) is not cyclic, since there are no elements of order 4. One should also observe that \( (\mathbb{Z}/8\mathbb{Z})^* \) and \( \mathbb{Z}/4\mathbb{Z} \) are, up to isomorphism, the only 4 element groups.