Solutions to Selected Problems on HW#5

Note: In pretty much any mathematical problem beyond the level of high school algebra, there will be many possible solutions and many ways to elegantly write up the solution. So while your solution need not look exactly like what’s here, it should be direct and clearly written.

23 (pg. 95):
(a) False: \((1 \, 2)(3 \, 4) \in S_4\) is most certainly not a cycle.
(b) True: By definition, a cycle is merely a particular kind of permutation.
(c) Neither: This is a really subjective question: One can certainly intuitively define the sign of a permutation, either through linear algebra, or through suspension of disbelief that a non-unique decomposition into transpositions won’t have crazy parity. But some more formally-minded readers may not like seeing a definition until the properties of the underlying objects have been sufficiently specified. So it’s up to you...
(d) False: Consider \(H := \langle (1 \, 2)(3 \, 4)(5 \, 6) \rangle\). The generator of \(H\) is clearly an odd permutation, so \(H\) satisfies the hypothesis. However, \(H\) has just 2 elements \((e\) and \((1 \, 2)(3 \, 4)(5 \, 6))\), neither of which is a transposition.
(e) False: \(A_5\) is an index 2 subgroup of \(S_5\) and the latter has 120 elements. So \(A_5\) actually has 60 elements.
(f) False: \(S_1\) and \(S_2\) are cyclic.
(g) True: The quickest way out is to observe that \(A_3\) has cardinality \(\frac{3!}{2} = 3\) and is thus cyclic (thanks to Lagrange’s Theorem).
(h) True: In fact, an isomorphism from the stated subgroup to \(S_7\) is given by the map sending
\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8
\end{bmatrix}
\mapsto
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7
\end{bmatrix}.
\]

(i) True: In fact, an isomorphism from the stated subgroup to \(S_7\) is given by the map sending
\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8
\end{bmatrix}
\mapsto
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
g(a_1) & g(a_2) & g(a_3) & g(a_4) & g(a_5) & g(a_6) & g(a_7)
\end{bmatrix},
\]
where \(g: \{1, \ldots, 8\} \rightarrow \{1, \ldots, 7\}\) is the map defined by setting \(g(x)\) to \(x\) or \(x - 1\), according as \(x < 5\) or \(x > 5\).
(j) False: \(e\) is an even permutation and thus the putative subgroup would be missing an identity element. Not cool.
4 (pg. 101):  
For a cyclic group (like \( \mathbb{Z}/12\mathbb{Z} \) is), this is most easily done by considering translations of the subgroup by the generator. In other words, consider \( \langle 4 \rangle, 1 + \langle 4 \rangle, 2 + \langle 4 \rangle, \) and \( 3 + \langle 4 \rangle. \) (Note that we stopped at 3 because 4 \( \in \langle 4 \rangle. \)) More explicitly, we obtain:

\[
\begin{align*}
\{0, 4, 8\} \\
\{1, 5, 9\} \\
\{2, 6, 10\} \\
\{3, 7, 11\}.
\end{align*}
\]

Now we merely observe that the preceding sets partition \( \mathbb{Z}/12\mathbb{Z} \), so we have indeed found all the left cosets. Note also that since \( \mathbb{Z}/12\mathbb{Z} \) is cyclic, \( \mathbb{Z}/12\mathbb{Z} \) is Abelian. So the left cosets and right cosets are the same, and we’re done.

15 (pg. 102):  
Since we are dealing with the a cyclic subgroup, it is best to first find the order of the generator. This is easiest when the generator is written as a product of disjoint cycles, so let’s do this first: expanding the given expression for \( \sigma \), we see that \( \sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{bmatrix}. \)

Following the orbit of 1 (and the orbits of any future elements missing subsequent orbits), we then obtain the decomposition \( \sigma = (1 2 3 5 4). \) In other words, \( \sigma \) is a 5-cycle, and thus has order 5. By Lagrange’s Theorem, \( \langle \sigma \rangle \) thus has index \( \frac{5!}{5} = 24. \)