On the Multilinearity of the Determinant

Clara Purdue was asking about the multilinearity of the determinant, so I thought I would say a few things here. (Although this is covered to some extent in Chapter 2 of Leon, I may have neglected to cover this in lecture.) The multilinearity of the determinant can be summarized quickly as the following property of det:

\textbf{Lemma 1} Let $F$ be any field and $A \in F^{n \times n}$. Suppose, for all $i \in \{1, \ldots, n\}$, that the $i^{th}$ row of $A$ is $\alpha_i$ and the $i^{th}$ column of $A$ is $a_i$. Also let $a'_1 \in F^{n \times 1}$ and $\alpha'_1 \in F^{1 \times n}$ and $\gamma, \delta \in F$. Then

$$\det[\gamma a_1 + \delta a'_1, a_2, \ldots, a_n] = \gamma \det[a_1, a_2, \ldots, a_n] + \delta \det[a'_1, a_2, \ldots, a_n]$$

and

$$\det \begin{bmatrix} \gamma \alpha_1 + \delta \alpha'_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \gamma \det \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} + \delta \det \begin{bmatrix} \alpha'_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$.

Thanks to the alternating property of the determinant with respect to swapping rows and columns, a similar statement can be made for any fixed column (resp. row) that is a linear combination of other column (resp. row) vectors.

As a simple example, observe that

$$\det \begin{bmatrix} x+1 & 1 \\ 1 & x+1 \end{bmatrix} = x \det \begin{bmatrix} 1 & 1 \\ 0 & x+1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & x+1 \end{bmatrix}$$

if we apply multilinearity to the first column of the left-hand matrix above. If we then apply multilinearity to the second columns of both the matrices on the right-hand side above, then we get:

$$\det \begin{bmatrix} x+1 & 1 \\ 1 & x+1 \end{bmatrix} = x \left( x \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) + \left( x \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$= x^2 \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + x \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + x \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= x^2 + 2x$$

While it may be obvious that the $2 \times 2$ determinant above can be computed more efficiently other ways (e.g., via the alternating sum of products expansion), multilinearity turns out to be much more useful for certain larger, structured matrices...