Some practice problems...

J. Maurice Rojas

August 11, 2014

1. Suppose $A, B \in \mathbb{C}^{n \times n}$ and let us denote the trace of $A$ by $\text{Tr}(A)$. Is it always true that $\text{Tr}(AB) = \text{Tr}(A) \text{Tr}(B)$? Please either prove or give a counter-example.

This one turns out to be false!: Perhaps the simplest counter-example comes from setting $A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We clearly obtain $\text{Tr}(AB) = 0$ but $\text{Tr}(A) = \text{Tr}(B) = 1$. ■

This partially corrects a misstatement I made in class on Monday, Aug. 11. The next problem completes the correction...

2. Suppose $A, B \in \mathbb{C}^{n \times n}$. Please prove that $\text{Tr}(AB) = \text{Tr}(BA)$.

We proceed directly: Assuming $A = [a_{i,j}]$ and $B = [b_{i,j}]$, the $(i,i)$-entry of $AB$ is simply $\sum_{\ell=1}^{n} a_{i,\ell} b_{\ell,i}$. Similarly, the $(i,i)$-entry of $BA$ is $\sum_{\ell=1}^{n} b_{i,\ell} a_{\ell,i}$. So then,

\[
\text{Tr}(AB) = \sum_{k=1}^{n} \sum_{\ell=1}^{n} a_{k,\ell} b_{\ell,k} = \sum_{\ell=1}^{n} \sum_{k=1}^{n} a_{k,\ell} b_{\ell,k} = \sum_{\ell=1}^{n} \sum_{k=1}^{n} b_{\ell,k} a_{k,\ell} = \text{Tr}(BA).
\]

(So the whole proof boils down to merely changing the order of a double-summation!) ■

In particular, our proof on Monday, Aug. 11, that $ST$ and $TS$ have the same characteristic polynomial still goes through: The coefficients still match up, just via the equality proved in this problem instead...

3. Please find bases for the right nullspaces of the following matrices:

(a) $\begin{bmatrix} 1 & 0 & 2 & 4 & 5 & 6 \\ 0 & 1 & 3 & 5 & 7 & 9 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 11 & 2 \\ 0 & 0 & 0 & 1 & -5 & 3 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 21 & 7 \\ 1 & 0 & 0 & 0 & 1 & -15 & 3 \end{bmatrix}$,

(d) $\begin{bmatrix} 4 & 3 & 0 & 21 & 7 \\ 2 & 5 & 1 & 7 & 3 \end{bmatrix}$.

4. Please find 2 distinct matrices $M$ satisfying both $M \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $M \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
First note that $M$ must be $3 \times 3$. There are then at least two ways to proceed...

**Solution #1:** We can simply proceed as suggested in class a few ways ago: Use elementary column operations on both sides. More precisely, observe that we can compress the two desired equalities into a single matrix equality: 

$$
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
4 & 5 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
.$$ 

Then, we can simply stack the matrix on the left-side atop the matrix on the right-side, and then start performing elementary column operations to simplify matters:

$$
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
0 & 1 \\
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
1 & 2 \\
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
.$$ 

For example, if we choose $v$ and $v$ see that one matrix that works is $M$ and $M$ work. If we then assume two (distinct) easy extra conditions, say, (a) $M = 
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{bmatrix}$, we easily see that the first two columns depend only $(c, f, i)$: We must have $M = 
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{bmatrix}$ and $M = 
\begin{bmatrix}
2 - 3c \\
-1 - 3f \\
0 - 3i \\
\end{bmatrix}$. So another $M$ could be, for instance, $M = 
\begin{bmatrix}
-1 & -1 & 1 \\
2 & -1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}$ (taking $c=1$ and $f=i=0$).

**Solution #2:** Since the two vectors we are trying to map to new values are linearly independent, we could just pick a new vector $v$ (linearly independent from the first two) and send it to some other random vector. We can then conclude simply via matrix division, but we must check that the our chosen $v$ is indeed linearly independent from the first two.

For example, if we choose $v = 
\begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}$, we see that $v$ can work. If we then assume two (distinct) easy extra conditions, say, (a) $M = 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}$ or (b) $M = 
\begin{bmatrix}
1 \\
0 \\
1 \\
\end{bmatrix}$, we see that we can take $M = 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}$ for (a) or $M = 
\begin{bmatrix}
1 & 2 & 0 \\
2 & 3 & 0 \\
4 & 5 & 1 \\
\end{bmatrix}$ for (b). I would probably accept an answer in this form, and forgive you for not computing the matrix inverse.
5. Assume, for some unknown complex constants $a, b, c, d, e, f$ that we have 
\[
\begin{pmatrix}
1 & a & b \\
1 & c & d \\
1 & e & f \\
\end{pmatrix}
\]
and 
\[
\begin{pmatrix}
1 & a & b \\
2 & c & d \\
3 & e & f \\
\end{pmatrix}
\]
are such that 
\[
\det
\begin{pmatrix}
1 & a & b \\
1 & c & d \\
1 & e & f \\
\end{pmatrix}
= 7 
\quad \text{and} \quad 
\det
\begin{pmatrix}
1 & a & b \\
2 & c & d \\
3 & e & f \\
\end{pmatrix}
= 11,
\]
please compute 
\[
\det
\begin{pmatrix}
1 & a & b \\
-3 & c & d \\
-5 & e & f \\
\end{pmatrix}
.
\]

6. Please find a Schur decomposition for the following matrices:
(a) 
\[
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
\end{pmatrix}
, 
\quad \text{(b)} \quad
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
, 
\quad \text{(c)} \quad
\begin{pmatrix}
7 & 7 \\
7 & 7 \\
\end{pmatrix}
, 
\quad \text{(d)} \quad
\begin{pmatrix}
2 & 3 \\
5 & 11 \\
\end{pmatrix}
,
\]
(e) 
\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \\
\end{pmatrix}
\]
(as a function of $\theta$),
(f) 
\[
\begin{pmatrix}
7 & 0 & 0 & 7 \\
0 & 2 & 3 & 0 \\
0 & 5 & 11 & 0 \\
7 & 0 & 0 & 7 \\
\end{pmatrix}
.
\]

7. If $S \in \mathbb{C}^{n \times n}$ is invertible, and $T \in \mathbb{C}^{n \times n}$, do $T$ and $STS^{-1}$ always have the same eigenvectors? Please prove or give a counter-example.

8. Suppose $A \in \mathbb{R}^{n \times n}$ is orthogonal. Prove that $\det A \in \{\pm 1\}$.

9. Please prove that if $A \in \mathbb{R}^{n \times n}$ is orthogonal and triangular, then it must be diagonal. Please also say exactly what kind of elements can occur on the diagonal (and prove that this is so).

10. Please prove that the matrix 
\[
A :=
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\in \mathbb{R}^{n \times n}
\]

satisfies $A^n = O$. 

11. Please find all eigenvalues and eigenspaces for the $n \times n$ matrix with all entries equal to 1.

12. For the matrix $\frac{vv^T}{v^Tv}$ (with $v \in \mathbb{R}^n$ a nonzero column vector), please find all the eigenvalues. Can you find a simple description of all its eigenspaces?

13. For the matrix $I - \frac{vv^T}{v^Tv}$ (with $v \in \mathbb{R}^n$ a nonzero column vector and $I$ the $n \times n$ identity matrix), please find all the eigenvalues. Can you find a simple description of all its eigenspaces?

14. Please prove that, for any matrices $S, T \in \mathbb{C}^{n \times n}$ with $S$ invertible, $ST$ and $TS$ have the same eigenvalues.

15. Please prove that, for any matrices $S, T \in \mathbb{C}^{n \times n}$, $ST$ and $TS$ have the same eigenvalues.