The Connectivity of $SO(n)$

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September 18, 2013

Abstract

We give a proof, based on unitary diagonalization, that the (real) special orthogonal group $SO(n)$ is path-connected.

Recall that an $n \times n$ *unitary* matrix is a matrix $U \in \mathbb{C}^{n\times n}$ satisfying $UU^* = U^*U = I_n$ where $(\cdot)^*$ denotes conjugate transpose and I_n is the $n \times n$ identity matrix. Recall also that $SO(n)$ is the group of all *real* unitary $n \times n$ matrices with determinant 1. We begin with a basic linear algebra fact, following easily from a classical theorem of Schur (see, e.g., [Pra94, pg. 86]).

Lemma 0.1 *Given any unitary matrix U we can find a unitary matrix V such that $\Lambda := V^*UV$ is diagonal and all the diagonal entries of Λ have absolute value 1. In particular, U is diagonalizable, its eigenvectors are (unitarily) orthogonal, and all its eigenvalues have norm 1.* ■

Corollary 0.2 *Given any matrix $M \in SO(n)$, there is a real orthogonal $n \times n$ matrix A such that $A^\top MA$ is block-diagonal, with each block either a 2×2 rotation matrix or 1.*

Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is actually the matrix corresponding to rotating $\mathbb{R}^2$ by an angle of π radians. So we can fact have -1s along the diagonal of the matrix asserted above, but their number must be *even*.

Proof of Corollary 0.2: Let Λ be the diagonal matrix consisting of the eigenvalues of M and V the corresponding matrix of eigenvectors. By Lemma 0.1, we see that $\Lambda = V^*MV$. Now note that the characteristic polynomial of M has all its coefficients real. This implies that the non-real eigenvalues of M come in conjugate pairs. Clearly then, we can order the entries of Λ from upper-left to lower-right so that the non-real eigenvalues come in conjugate pairs, followed by any -1s, then followed by any 1s. Similarly, we may assume that the eigenvectors come in the corresponding order.

Let $\{\lambda, \bar{\lambda}\}$ be any conjugate pair of non-real eigenvalues. Since M is real, the corresponding eigenvectors must be of the form $\{v, \bar{v}\}$, i.e., each coordinate of one eigenvector is the conjugate of the corresponding coordinate of the other.

Now observe that $w_+ := (v + \bar{v})/\sqrt{2}$ and $w_- := (v - \bar{v})/\sqrt{-2}$ are real vectors that are linearly independent and generate (over $\mathbb{C}$) the same subspace as $\{v, \bar{v}\}$. In particular, $[w_+, w_-] = [v, \bar{v}]\, S$, where

$$S := \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{-2} \\ 1/\sqrt{2} & -1/\sqrt{-2} \end{bmatrix}.$$

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Note also that S is unitary and

$$S^* \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} S = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for some real angle $\theta \in (0, 2\pi)$. Letting Σ denote the $n \times n$ block-diagonal matrix whose blocks are either 1 or S (corresponding to eigenvalues of M that are 1 or not 1), we then see that $\Sigma\Lambda\Sigma^*$ is the block-diagonal matrix we seek. Furthermore, by our earlier application of Lemma 0.1, our desired block-diagonal matrix is also equal to $\Sigma^* V^* M V \Sigma$. Note in particular that $V\Sigma$ is orthogonal since it is real, as well as a product of unitary matrices. So we can take $A = V\Sigma$. ■

We can now prove our desired connectivity result very quickly.

Theorem 0.3 *The (real) special orthogonal group $SO(n)$ is path-connected.*

Proof: It will clearly suffice to show that any $M \in SO(n)$ can be connected by a path in $SO(n)$ to the $n \times n$ identity matrix. In particular, by Corollary 0.2, we can rewrite $M = ARA^\top$ where A is orthogonal and R is a block-diagonal matrix with each block either a 2×2 rotation matrix or 1.

Suppose the underlying rotation angles are $\theta_1, \ldots, \theta_{n'}$. Clearly, $n' \leq n/2$. We now construct our path as follows: Let $R(t)$ denote the matrix obtained by replacing θ_i by $(1-t)\theta_i$ in R. Finally, let $M(t) := AR(t)A^\top$. Clearly, $M(0) = M$ and $M(1) = I_n$. Furthermore, the entries of $M(t)$ are clearly continuous functions of t. So we are done. ■

References

[Pra94] Prasolov, V. V., *Problems and Theorems in Linear Algebra,* translations of mathematical monographs, vol. 134, AMS Press, 1994.