Abstract

Sturm’s Theorem is a fundamental 19th century result relating the number of real roots of a polynomial \( f \) in an interval to the number of sign alternations in a sequence of polynomial division-like calculations. We provide a short direct proof of Sturm’s Theorem, including the numerically vexing case (ignored in many published accounts) where an interval endpoint is a root of \( f \).

1 Introduction

Counting the number of roots of a polynomial in an interval is a fundamental algorithmic problem in real algebraic geometry, and forms the core of techniques for deeper problems such as real-solving and the first order theory of the reals. On the practical side, numerous problems from control theory and physical modeling reduce to solving systems of polynomial equations over the real numbers, and one can not solve a system numerically until one understands how to count the number of roots in an interval.

We will present, from scratch, a strengthened version of a classical result of Sturm. To begin, let us review some basic ideas.

Notation Recall that \( \mathbb{R}[x_1] \) is the collection of all polynomials in the variable \( x_1 \) with real coefficients. For any \( f \in \mathbb{R}[x_1] \) of degree \( d \) we then define its pseudo-remainder sequence (a.k.a. Sturm sequence) to be \( P_f := (p_0, \ldots, p_d) \), where \( p_0 := f \), \( p_1 := f’ \) (the derivative of \( f \)),
\[
p_i := q_{i+1} p_{i+1} - p_{i+2} \text{ for all } i \in \{0, \ldots, d-2\},
\]
and \( q_{i+1} \) and \( -p_{i+2} \) are respectively the quotient and remainder obtained from dividing \( p_i \) by \( p_{i+1} \). We also define \( P_f(c) := (p_0(c), \ldots, p_d(c)) \) for any \( c \in \mathbb{R} \) and \( V_f(c) \) to be the number of sign alternations in the sequence \( P_f(c) \). In particular, the number of sign alternations in an arbitrary sequence \( (s_0, \ldots, s_d) \) is simply the number of \( j \in \{0, \ldots, d-1\} \) such that there is a \( k > 0 \) with \( s_j s_{j+k} < 0 \) and \( s_{j+\ell} = 0 \) for all \( \ell \) with \( j < \ell < j+k \). Finally, we let \( \sigma : \mathbb{R} \rightarrow \{-1, 0, 1\} \) be the sign function, which maps all positive (resp. negative) numbers to 1 (resp. \(-1\)) and 0 to 0. We also naturally extend \( \sigma \) to sequences by \( \sigma(s) := (\sigma(s_0), \ldots, \sigma(s_d)) \).

Example 1.1 For \( f(x_1) := x_1^4 - 2x_1^2 + 1 \), we clearly obtain
\[
P_f(x_1) = (x_1^4 - 2x_1^2 + 1, 4x_1^3 - 4x_1, x_1^2 - 1, 0, 0),
\]
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\(\sigma(P_f(-2)) = (1, -1, 1), \ \sigma(P_f(0)) = (1, 0, -1),\) and thus \(V_f(-2) = 2\) and \(V_f(0) = 1.\) Note also that \(-1\) is the only root of \(f\) in the half-open interval \((-2, 1)\), and that Sturm sequences can terminate with more than one zero term. ♦

Recall that a root \(\zeta\) of \(f\) is a multiple (or degenerate) root iff \(f(\zeta) = f'(\zeta) = 0.\) Roots \(\zeta\) with \(f'(\zeta) \neq 0\) are usually called simple or non-degenerate.

**Theorem 1.2 (Refined Sturm’s Theorem)** For any \(f \in \mathbb{R}[x_1] \setminus \{0\}\) and any real numbers \(a\) and \(b\) with \(a \leq b,\) let \(N_f(a, b)\) denote the number\(^1\) of roots of \(f\) in the half-open interval \((a, b)\).\(^2\) Then

1. If neither \(a\) nor \(b\) are multiple roots, then \(N_f(a, b) = V_f(a) - V_f(b).\)
2. If \(f(c) = f'(c) = 0\) then \(V_f(c) = 0.\)

Note in particular that we can count roots in \((a, b)\) even when one or both endpoints are roots of \(f\) — provided no multiple roots occur at either endpoint. Assertion (2), while almost trivial to prove, is the main reason one needs to avoid multiple roots at end-points when using Sturm’s Theorem: The information carried by the sign alternations of \(P_f(c)\) is lost entirely when \(c\) is a multiple root.

**Example 1.3** It is easily checked that the roots of \(f(x_1) := x_1^3 - 5x_1^2 + 7x_1 - 3\) are precisely \((2, 3),\) with 2 a multiple root and 1 a simple root, and that

\[P_f(x_1) = (x_1^3 - 5x_1^2 + 7x_1 - 3, 3x_1^2 - 10x_1 + 7, \frac{8}{3}x_1 - \frac{8}{9}, 0).\]

Clearly then, \(V_f(1) = 0, V_f(2) = 1,\) and \(V_f(3) = 0,\) so Sturm’s Theorem is confirmed for the interval \((2, 3)\). However, we also see that Sturm’s Theorem cannot be applied to the interval \((1, 3)\) since \(V_f(1) - V_f(3) = 0\) and \(f\) in fact still has a root in \((1, 3)\). ♦

Curiously, most published accounts of Sturm’s Theorem avoid considering the presence of any kind of root at an endpoint. Furthermore, many accounts assume that \(f\) has only simple roots. In practice, such an assumption can only be enforced by computing square-free parts — a potentially wasteful (and numerically unstable) computation, especially when speed is critical.

The proof of Sturm’s Theorem is elementary, but is frequently derived as a consequence of more intricate constructions. Considering its deep importance in numerical software, we present a short direct proof. The theorem follows easily from a single lemma.

**Lemma 1.4** Suppose \(f \in \mathbb{R}[x_1]\) has positive degree \(d,\) and \(c \in \mathbb{R}.\) Then the following properties hold for the Sturm sequence \(P_f(c) = (p_0(c), \ldots, p_d(c))):$

1. (Sign Alternation Over a Simple Root) Suppose \(p_0(c) = 0\) and \(p_1(c) \neq 0.\) Then for all \(\epsilon > 0\) sufficiently small, \(\sigma(p_0(c - \epsilon)) = -\sigma(p_1(c - \epsilon))\) and \(\sigma(p_0(c + \epsilon)) = \sigma(p_1(c + \epsilon)).\)

2. (Stability Under Common Multiples) Suppose \(p_0\) and \(p_1\) are each divisible by \(g \in \mathbb{R}[x_1].\) Then \(P_f = \frac{1}{g}P_{f/g}.$

\(^1\)This theorem counts distinct roots and thus does not count multiple roots more than once.

\(^2\)When \(a < b\) this half-open interval includes \(b\) but does not include \(a,\) and we use the convention \((a, a) = \emptyset.\) European authors frequently use \([a, b]\) for what we call \((a, b).\)
3. (Matching Sign Flips Over a Simple Node) Suppose \( p_i(c) = 0 \) and \( p_{i+1}(c) \neq 0 \) for some \( i \geq 1 \). Then \( \sigma(p_{i-1}(c)) = -\sigma(p_{i+1}(c)) \). Furthermore, for all \( \varepsilon > 0 \) sufficiently small, \( \sigma(p_{i-1}(c - \varepsilon)) = -\sigma(p_{i+1}(c - \varepsilon)) \) and \( \sigma(p_{i-1}(c + \varepsilon)) = -\sigma(p_{i+1}(c + \varepsilon)) \).

4. (Propagation of Zeroes) For any \( i \in \{0, \ldots, d - 2\} \), \( p_i(c) = p_{i+1}(c) = 0 \) implies that \( p_j(c) = 0 \) for all \( j \geq i + 2 \).

We are now ready to prove our refined version of Sturm’s Theorem.

Proof of Theorem 1.2:

**Assertion (2):** Since \( p_0(c) = f(c) \) and \( p_1(c) = f'(c) \), the recurrence defining the Sturm sequence immediately implies that \( p_j(c) = 0 \) for all \( j \geq 2 \), so we are done.

**Assertion (1):** If \( a = b \), or \( f \) is a nonzero constant, then we clearly have \( N_f(a, b) = V_f(a) - V_f(b) = 0 \) and Assertion (1) indeed holds. So let us assume \( a < b \) and that \( f \) has positive degree \( d \).

Let us now reduce to the special case where the following condition holds:

\[ (a, b) \text{ contains at most 1 root of } f, \text{ with } b \text{ non-degenerate if it is a root of } f \]

To do so, suppose first that \( f \) has roots in \((a, b) \) and that, in strictly increasing order, they are exactly \( \zeta_1, \ldots, \zeta_m \). Letting \( (c_1, \ldots, c_m) \) be any sequence satisfying

\[ a < c_1 < \zeta_1 < c_2 < \zeta_2 < \cdots < c_m < \zeta_m \leq b, \]

we then see that \( V_f(a) - V_f(b) \) is exactly

\[
(V_f(a) - V_f(c_1)) + (V_f(c_1) - V_f(c_2)) + \cdots + (V_f(c_{m-1}) - V_f(c_m)) + (V_f(c_m) - V_f(b)).
\]

Since, by definition, \( N_f(a, b) \) is exactly

\[ N_f(a, c_1) + N_f(c_1, c_2) + \cdots + N_f(c_{m-1}, c_m) + N_f(c_m, b) \]

it then clearly suffices to prove \( N_f(a, c_1) = V_f(a) - V_f(c_1), \) \( N_f(c_1, c_2) = V_f(c_1) - V_f(c_2), \) \( \ldots, N_f(c_{m-1}, c_m) = V_f(c_{m-1}) - V_f(c_m), \) and \( N_f(c_m, b) = V_f(c_m) - V_f(b) \). In other words, whether or not \( f \) has roots in \((a, b) \), we can indeed assume Condition (\( \ast \)).

Let us now reduce even further to the special case where the following slightly stronger condition holds:

\[ (a, b) \text{ contains at most 1 root of } f, \text{ and any root of } f \text{ in } (a, b) \text{ is simple.} \]

To do so, observe that \( g := \gcd(f, f') \in \mathbb{R}[x_i] \) and, for all \( i \in \{1, \ldots, m\} \), \( f/g \) is divisible by \( x - \zeta_i \) but not divisible by \( (x - \zeta_i)^2 \). (The latter fact follows easily from the product rule for differentiation.) So \( f \) and \( f/g \) have the same real roots, except that all the real roots of \( f/g \) are simple. By Assertion (2) of Lemma 1.4 we then obtain \( \sigma(P_f(c)g(c)) = \sigma(P_{f/g}(c)) \) and thus \( \sigma(P_f(c)) = \sigma(g(c))\sigma(P_{f/g}(c)) \) for all real \( c \). In particular, \( V_f(c) = V_{f/g}(c) \) as long as \( c \) is not a multiple root. Since we are assuming that neither \( a \) nor \( b \) are multiple roots, we can then clearly assume (\( \ast \)).

We are now nearly done: Thanks to Condition (\( \ast \)), the case where \( (a, b) \) contains a unique root \( \zeta \) follows immediately from Assertion (1) of Lemma 1.4, assuming we have proved the case where \( (a, b) \) contains no roots of \( f \). For then we obtain

\[ N_f(a, b) = N_f(a, \zeta - \varepsilon) + N_f(\zeta - \varepsilon, \zeta + \varepsilon) + N_f(\zeta + \varepsilon, b) = 0 + 1 + 0, \]

where \( \varepsilon > 0 \) is sufficiently small.

The final case where \( (a, b) \) has no roots requires only one more refinement: In our preceding subdivision used to enforce Condition (\( \ast \)), suppose we picked more \( c_i \), so that the roots
of \( p_1, \ldots, p_d \) (as well as those of \( p_0 = f \)) were also interlaced. Via the same trick of cancellations in an alternating sum, we can thus reduce even further to the special case where \((a, b]\) contains no roots of \( f \) and contains at most 1 root of \( p_1 \cdots p_d \). By Assertion (3) of Lemma 1.4, we are done. ■

**Proof of Lemma 1.4:** Assertions (2) and (4) follow immediately from the recurrence defining the Sturm sequence. Indeed, upon noting that every \( p_i \) is a polynomial linear combination of \( p_0 \) and \( p_1 \), it is clear that \( g | p_0 \) and \( g | p_1 \) together imply that every \( p_i \) is divisible by \( g \). Assertion (2) then follows immediately from the uniqueness of remainders in polynomial division. For Assertion (4) one merely proceeds by induction.

To prove Assertion (1) note that if \( f'(c) = p_1(c) > 0 \) (resp. \( f'(c) = p_1(c) < 0 \)) then \( f = p_0 \) is locally increasing (resp. decreasing) at \( c \). Since the sign of \( p_1 \) is locally constant at \( c \), Assertion (1) follows immediately.

Assertion (3) follows easily upon observing that \( p_i(c) = 0 \) implies that \( p_{i-1}(c) = -p_{i+1}(c) \), thanks to the recurrence defining the Sturm sequence. In particular, one need only observe that the signs of \( p_{i-1} \) and \( p_{i+1} \) are locally constant at \( c \), so \( \sigma(p_{i-1}(c - \varepsilon)) = \sigma(p_{i-1}(c)) = \sigma(p_{i-1}(c + \varepsilon)) \) and \( \sigma(p_{i+1}(c - \varepsilon)) = \sigma(p_{i+1}(c)) = \sigma(p_{i+1}(c + \varepsilon)) \). ■

**Acknowledgements**

We are most grateful for Jerry Friesen’s constant support of this project.

**References**

